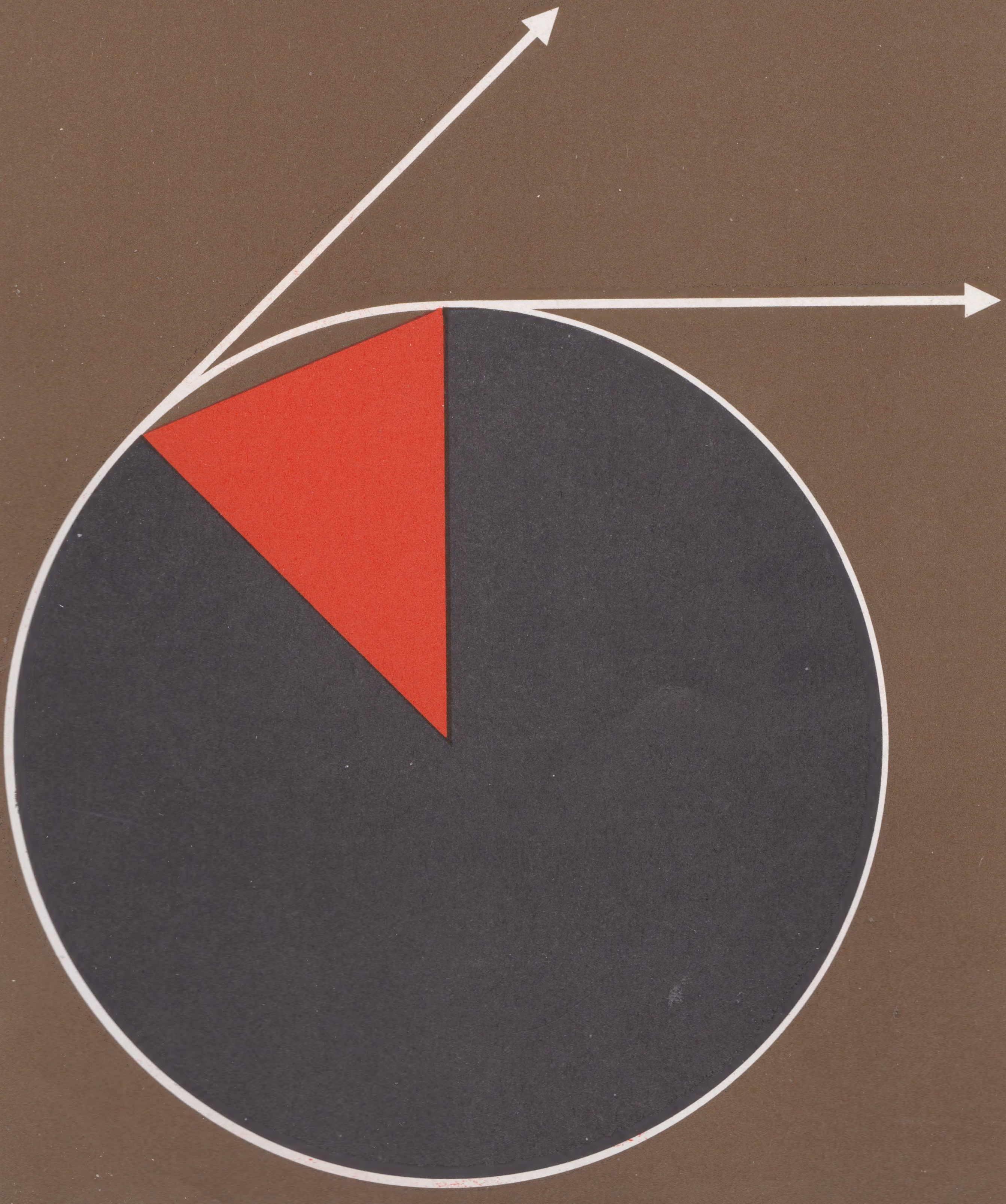
S 100 3 and 4

THE OPEN UNIVERSITY

Science Foundation Course Units 3 and 4

Mass, Length and Time
Forces, Fields and Energy



The Open University

Science Foundation Course Unit 3

MASS, LENGTH AND TIME

Prepared by the Science Foundation Course Team

THE OPEN UNIVERSITY PRESS

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The following people acted as consultants for certain components of the course:

J. D. Beckman
B. S. Cox
G. Davies
G. Holister
R. J. Knight
D. J. Miller
M. W. Neil
C. Newey
J. R. Ravetz
H. Rose

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1.5

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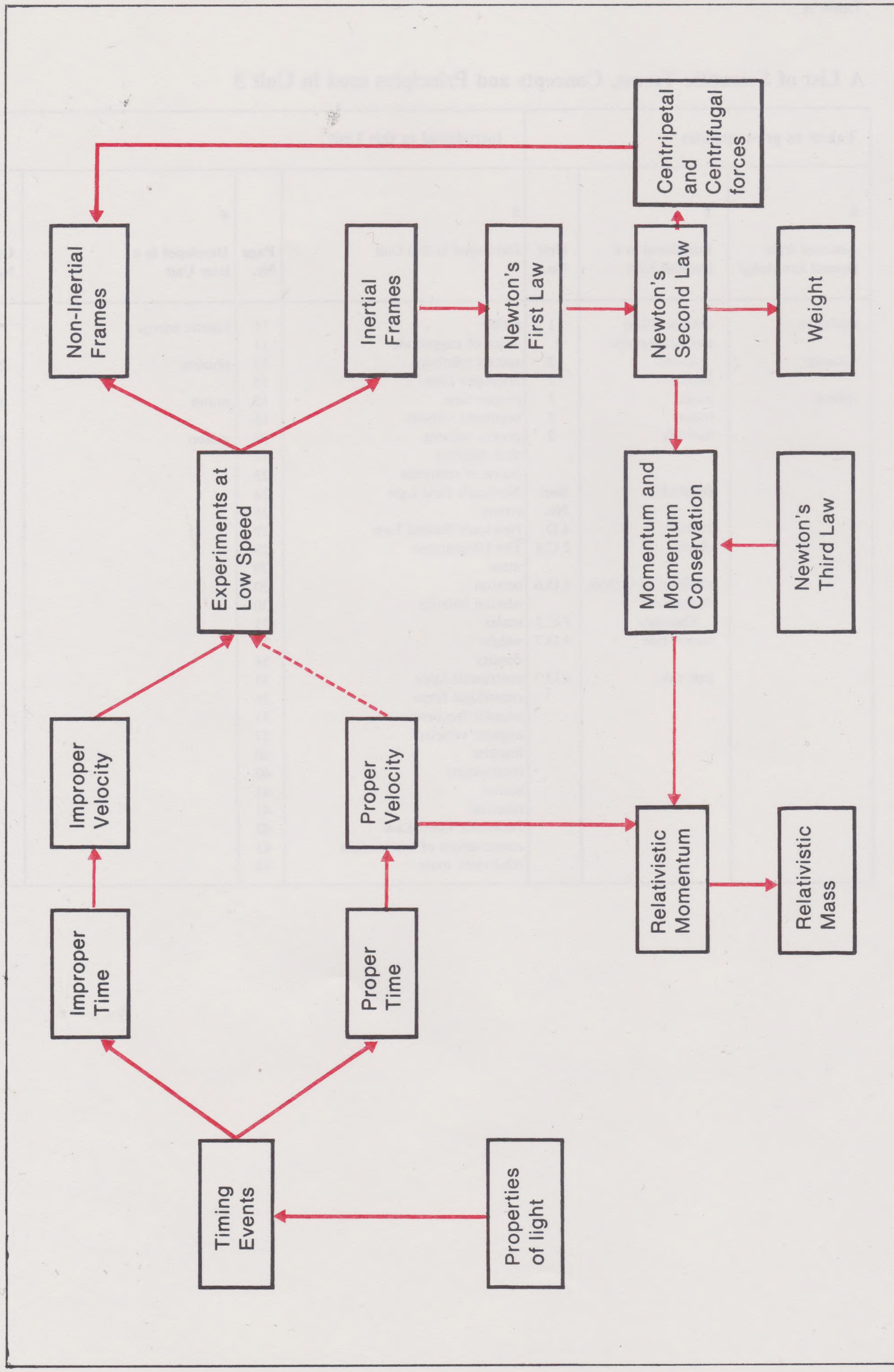


Properties of light
Timing Events
Improper Time
Proper Time
Improper Velocity
Proper Velocity
Experiments at Low Speed
Non-Inertial Frames
Inertial Frames
Newton's First Law
Newton's Second Law
Centripetal and Centrifugal forces
Weight
Momentum and Momentum Conservation
Newton's Third Law
Relativistic Momentum
Relativistic Mass

Objectives of Unit 3

When you have finished studying this Unit, you should be able to:

1 Define correctly, or recognize the best definition of, all the terms, concepts and principles in column 3 of Table A.

2 State those properties of light which are assumed in formulating the theory of special relativity. (*SAQ* 1)

3 Given appropriate data, to perform simple calculations relating proper time with improper time and proper velocity with improper velocity. (*SAQ* 2, 3, 4)

4 Perform calculations in elementary kinetics. (*SAQ* 6, 7, 8)

5 Perform simple calculations involving the application, with appropriate manipulation, of Newton's Laws of Motion. (*SAQ* 5, 9, 10 11, 12, 13, 14, 15, 16)

6 Define the momentum of a body in terms of its mass and its velocity and to be able to relate momentum with impulse. (*SAQ* 17, 18, 19, 20)

3.1 Introduction

In the first two Units we have pointed out some of the limitations of the human senses. We have also indicated ways in which man has attempted to extend their range. The word *attempt* is used deliberately here although, up to now, we may have given the impression that extrapolation is a pretty straightforward operation. You may, for example, have been quite satisfied with our suggested method of measuring the distance of a star (Unit 2); a method relying on an extrapolation of Earth-bound triangulation. However, we have many reservations about this technique. Only if light does travel in straight lines over large distances, only if the star itself does not move significantly during the year . . . only then are we prepared to assert that the star is so many metres away. Rather obvious qualifications, you may say. Agreed, but had we been educated within the ethos of nineteenth-century physics we might have had more serious reservations, reservations stemming from an 'intuitive' picture of how light 'ought' to behave. To begin with we shall, albeit tentatively, embrace some of these notions about the nature of light. We shall then find that our 'intuitive' picture is simply not correct. This in turn will force us to the conclusion that there is much more to seemingly obvious experiments, such as measuring the speed of a car, than we previously thought possible! It will, perhaps thankfully, emerge that there is little discrepancy between our everyday and our precise ideas of a car's velocity, *provided* that it is moving significantly slower than the velocity of light. So we will be able to discuss what we mean by velocity and how we can change the velocity of everyday objects, with, in most everyday situations, the knowledge that the qualifications are relatively unimportant. But we will have to keep the qualifications in mind. At times they will lead to very unexpected predictions.

Some of the results which we shall present may appear to run contrary to common sense. But do not forget that 'common sense' is founded on everyday experiences. In Unit 2 we showed just how limited are our senses. Therefore the rules of behaviour which applied in everyday life need not, *a priori*, be expected to hold true outside this limited range. So do not be unduly worried on meeting results which appear strange. With greater familiarity the strange becomes acceptable.

3.2 The Properties of Light

3.2.1 Collecting information

A great many experiments—some scientists would claim all—ultimately depend in one way or another on the properties of electromagnetic waves. If you attempt to measure the height of a flower with a metre stick, you must use your eyes to judge the position of the flower relative to the scale. You can use a notched rule and rely on your sense of touch if you like, but the notches must ultimately be calibrated against the wavelength of krypton light. Indeed the sense of touch itself depends on the interaction of the atoms of your fingers with the atoms of the flower. Interactions between atoms are electromagnetic. Whenever you attempt to set your watch correctly by looking at a clock or a TV screen, the correctness of the setting depends not only on the time that the light takes to travel from the screen to your eye, but on the length of time the information took in passing through the set's electronics, and on the time taken . . . right back to the moment when the hands of the studio clock were opposite particular marks on the clock face.

If electromagnetic waves were to travel at, say, 10 m s^{-1}, by how much would the setting of your watch differ from the studio clock? In answering this question, assume that information also travels through the wires in your TV set at 10 m s^{-1}.

As an example, a set at the Open University is about 30 miles from the Oxford transmitter, which is itself about 50 miles from the studio in London, so the signal is transmitted over a total distance of about 80 miles, i.e. $80 \times 1\,500 = 1.2 \times 10^5$ m, which would take a time of $1.2 \times 10^5/10$ s $= 1.2 \times 10^4$ s, which is over $3\frac{1}{2}$ hours! (In making this type of calculation it is often helpful actually to write down that velocity = distance/time and then to cross multiply to obtain the required quantity.) Your own answer to this artificial problem should at least emphasize the importance of having a fast carrier of information when standardizing equipment. Standardization is an essential part of any quantitative investigation and of course the faster the carrier the more reliable will be the readings from the equipment that is standardized.

3.2.2 The speed of light

Why employ electromagnetic waves in standardizing equipment? Why not use sound waves instead?

Which travel faster in air: light or sound waves?

If you do not already know the answer you might care to attempt the experiment of standing at a point equidistant from a radio set 'pipping' out the time and a TV set showing a clock face. However, as the sources of information are different, your conclusion should be suspect. The observation that in a thunderstorm the lightning flash reaches us before the thunderclap is possibly a more convincing demonstration of the answer.

Can you think of a method of making an *order of magnitude* determination of the speed of light?

By requesting an order of magnitude measurement we are only seeking an answer which is correct to the nearest power of ten. In the present context it is of little interest whether the speed of light is, say, 576 m s^{-1} or 532 m s^{-1}, but it is worth knowing whether it is 10^{-1}, 10^{0}, 10^{2}, etc. m s^{-1}. If you have a torch you can try to measure the speed of light by flicking on the switch and measuring the time the light beam takes to travel to, say, a distant house and then return to your eyes. Alternatively switch on a light in a darkened room. You may be interested to know that Galileo attempted to measure the speed of light by placing two observers, each equipped with lamps, several miles apart. One observer uncovered his lamp at the same time starting a clock. The second observer uncovered his lamp as soon as he saw the light from the first observer's lamp. Once the first observer saw the other's lamp he stopped his clock.

If you decide that the time interval is 'zero', think again. Put a lower limit to the time interval, remembering that there is a minimum time interval of which we are conscious (a phenomenon exploited in cine photography where the apparently steady image in fact changes about 30 times each second). Your final estimate of the speed of light, or rather your estimate of the lower limit of the speed of light, will probably look pretty insignificant beside the accepted value for the velocity of light in a vacuum (or, closely enough, in air) of 2.998×10^{8} m s^{-1}.

You plan to measure the speed of light by determining the time which elapses between a pulse (a short burst) of light passing two points 20 m apart. Such a pulse could be produced by a short duration spark. What sort of time intervals must the clock used in timing the pulse be able to measure?

Time intervals of $20/(3 \times 10^{8})$ s, i.e. of the order of 10^{-7} s, can be measured with a cathode-ray oscilloscope. You will see such an instrument being employed in measuring the speed of light in this Unit's TV programme.

3.2.3 Factors affecting the speed of light

What factors might influence the speed of light? In particular are there any which might increase the speed? If the speed of light can be increased, then not only will the rate of transmission and reception of data be accelerated but the very instruments which are employed (e.g. clocks) can be more accurately calibrated against the master instruments—the laboratory standards, as we call them.

The first experiment might be to measure the speed of light through a different medium, say water. Surface waves on a liquid are known to travel slower across, say, motor oil than paraffin. If such waves are a reasonable analogy for the way light travels, then we might expect the speed of light to depend on the medium through which it is propagated. The experiment proper can be done by measuring the time it takes for a pulse of light to travel through a glass tank containing the liquid. The results show that light travels slower in all media which transmit it than in vacuum: e.g. the speed is down by a factor of 1.33 in water, and by 1.0003 in air. So, ideally, all our information should be transmitted through a vacuum.

How does the measured speed of light depend on the speed of movement of the light source? To get a feel for possible effects you can again consider the analogous situation of ripples in water. We can even go further if we wish and tentatively adopt the nineteenth-century model that all matter, even outer space, is pervaded by a medium (traditionally called the *aether*) through which light is propagated as a wave motion. The model is only being adopted as a means of providing clues as to possible effects. If the effects are discovered it does not, of course, vindicate the model. The hypothesis that the moon is made of green cheese may be a useful one if it leads us to try the experiment of growing bacteria on moondust! If bacteria should grow it would not prove that the moon is made of green cheese!

aether

Does the speed at which ripples move across water depend on the speed of the source of the ripples? Sets of ripples can be generated by skimming a stone across water. If you prefer not to skim a stone, watch an insect flitting across the surface of a pond generating sets of ripples as it goes.

The answer, perhaps surprisingly, is that the speed of the ripples is independent of the speed of the source. But what effect does moving a light source have on the speed of the emitted light? If the speed of light were only, say, 10 m s^{-1}, it would be easy to suggest a realistic home experiment.

Suggest such an experiment.

You could repeat the experiment of bouncing a torch beam off a distant object, but this time you would move the hand holding the torch. Try the experiment if you like but we doubt whether you will be able to reach any firm conclusions. Compared to the speed of light your best efforts will look quite trivial. Perhaps the most convincing demonstration of the effect of source movement comes from experiments with sub-nuclear particles called π° mesons. You will learn more about this type of particle in Unit 32. For the time being all you need know about the particle is that it can decay emitting in the process two short 'bursts' of radiation (called *photons*). In one such experiment π° mesons travelling faster than 99.98 per cent of the speed of light were found, on decaying, to produce photons

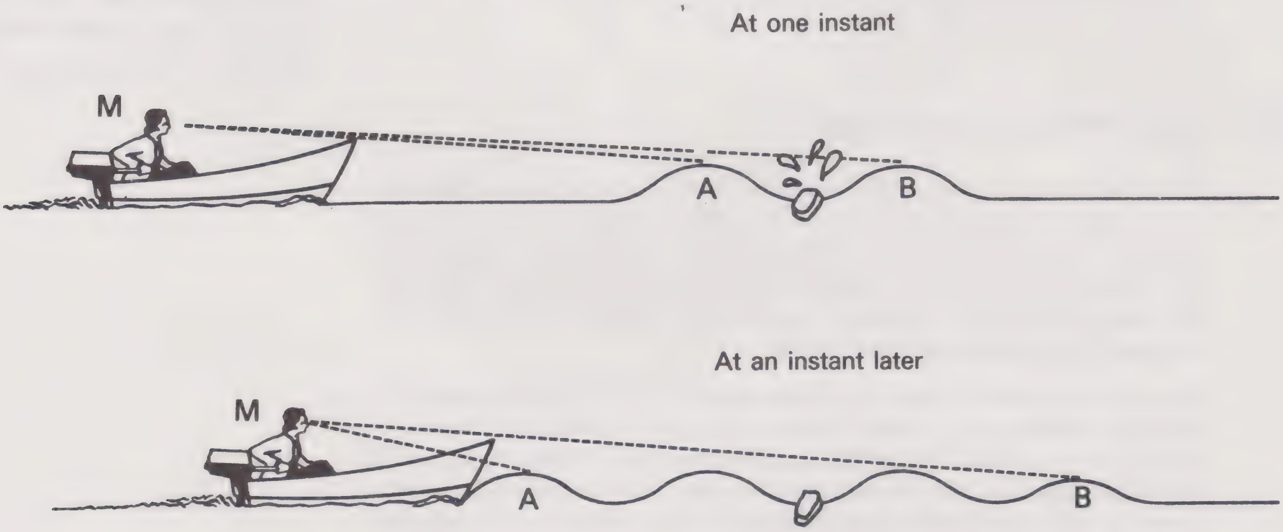


Figure 1 Showing how the apparent speed of water ripples depends on the speed of the observer. Compare the length MA in the upper drawing to the length MA in the lower one; do the same for the length MB. Satisfy yourself that MA changes much more than MB does.

moving at a measured speed of $2.9977 \pm .0004 \times 10^8$ m s^{-1}, which is in close agreement with the speed of light (2.9979×10^8 m s^{-1}) as measured from a stationary source.

Does the measured speed of water ripples depend on the speed of the observer?

It does. If you are crossing a pond towards the point where a stone has been dropped into the water, the apparent speed of the ripples coming towards you is increased; those receding from you appear to be slowed down (Fig. 1). You get the same effect if you are stationary, say on a bridge, but the water is streaming past below. What matters to an observer is his velocity relative to the medium carrying the waves.

Again, because of the high speed of light we cannot suggest realistic home experiments to look for the effect of observer movement. This remark is not, of course, intended to discourage you from running towards or away from a distant light source if you so wish! As the Earth itself is moving around the Sun at a speed of about 3×10^4 m s^{-1}, it might be more realistic to suggest experiments in which one observes a distant fixed star at six-monthly intervals; if the Earth is, say, moving towards the star in January then it will be receding from the star in July. So light from the star might

appear to move faster in January than in July. However, we need not expect such effects to be readily observed for even at speed of 3×10^4 m s^{-1} the Earth is moving ten thousand times slower than light. But there are instruments called optical interferometers which can easily detect changes of 1 part in 10^6 in the velocity of light. The phenomenon of optical interference, on which these instruments depend, will be discussed in Unit 28. In 1887 two Americans, Michelson and Morley, carried out an experiment which they hoped would demonstrate the seemingly 'obvious' fact that, as with water ripples, the measured velocity of light 'must' depend on the velocity of the Earth-bound interferometer. Surprisingly, Michelson and Morley failed to find the expected dependence; the velocity of light was independent of the velocity of the observer, i.e. of the interferometer. This negative result marked the demise of the aether model. If bacteria should fail to grow on moondust the experimenter might deduce that the moon is not made of green cheese, but a more pragmatic individual would simply note that bacteria does not grow on moondust. We will simply note that the measured velocity of light is independent of the velocity of the observer. Put differently, optical interference experiments (like those carried out by Michelson and Morley) give no indication that the laboratory, or, strictly speaking, the apparatus, is moving through space. Indeed all optical experiments have failed to give evidence of any uniform motion of the laboratory through space.

velocity of light independent of velocity of observer

To sum up:

1 The velocity of light depends on the medium through which it is travelling, having the highest value in a vacuum.

2 In any laboratory the speed of light is independent of the speed of its source.

3 Optical experiments performed within a laboratory (the Earth in our case) give no information about the uniform motion of that laboratory. We shall shortly show that the same is true for mechanical experiments, as indeed it is true for all type of experiments.

Such findings as these might well have come as a distinct surprise or even a shock to someone educated during the latter half of the nineteenth century. The last two findings in particular have had, as we shall see, profound consequences; they affect the way we perform the most 'obvious' of experiments. Indeed a distinct branch of physics known as *special relativity* has built up around these two findings. Much of the credit for this is due to Einstein.

special relativity

You may now attempt *SAQ* 1, p. 54, if you wish.

3.3 Some Predictions of Special Relativity

3.3.1 Proper and improper time

The techniques involved in measuring the speed of a car from first principles may seem so obvious as scarcely to warrant description. First you measure out some distance along a road, probably by 'stepping' it out with a metre stick. Next you measure the time which elapses between the car passing the two end points. Dividing the distance by the time interval gives the car's average speed. The difficulty arises in connection with the time interval. As we shall now prove, the driver of the car measures a different time interval from that measured by an observer stationary on the road!

If we are going to measure the time taken by a car to travel a certain distance, one of the problems is reaching agreement on the exact moment when the car passes the start and finish. The simplest way to do this is probably to take separate flash photographs and then compare them at leisure. No one is going to argue about a photograph.

Figure 2 illustrates a method for obtaining our two photographs. Mounted on the car next to a clock is a flash bulb S. A mirror M at distance L above the clock is also fixed to the car. A series of clocks identical with the one on the car are set out alongside the track. Before they were set out along the road these clocks were synchronized to read the same time when together. When the bulb S is fired, the clock on the car and the adjacent clock on the road are briefly illuminated. This event is photographed by a nearby camera. To avoid possible arguments, these two clocks, the camera, and the flash bulb are kept as close as possible, i.e. they are effectively at the same point in space. Once the photograph has been taken the flash of light continues up to the mirror, is reflected, and returns to illuminate yet again the same clock on the car and, in addition, whichever new clock on the road is adjacent to the one on the car. This new event is also photographed.

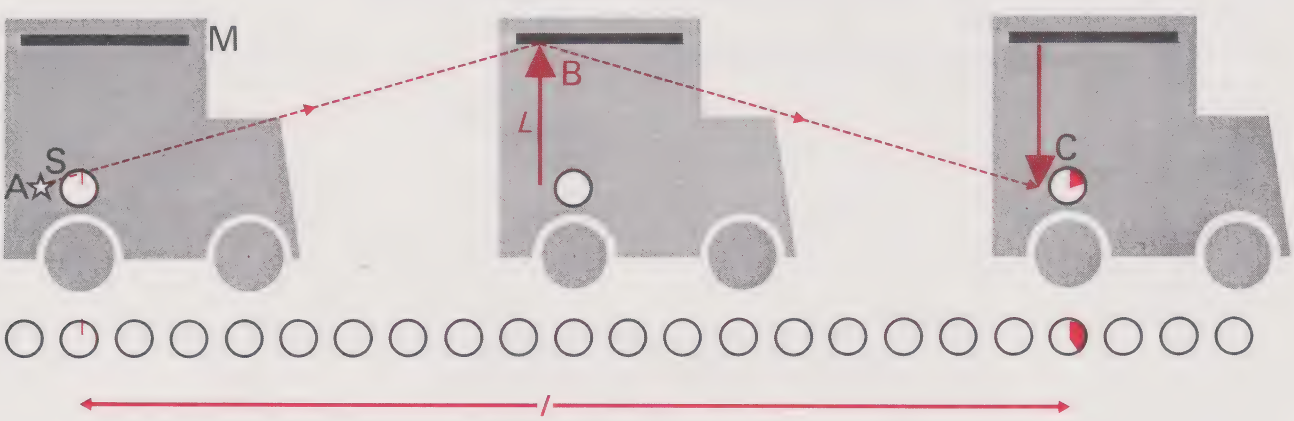


Figure 2 A technique for measuring the speed of a car. When the flash bulb S is fired it illuminates a clock on the car and an adjacent one alongside the road. Relative to the clocks set out along the road, the light flash subsequently follows path ABC, shown dashed. Relative to the clock on the car the light proceeds vertically up to mirror M, as shown in the full line. On returning down, the light flash illuminates yet again the clock on the car and whichever new clock on the road is adjacent to the one on the car. The horizontal scale in Figure 3 is, of course, very much larger than the vertical scale. (The speed of a motor car is a lot less than the speed of light!)

How will the clock readings compare in the two photographs?

So far as the clock on the car is concerned, the light beam travels a distance $2L$. Since light travels at a velocity c, the time taken for the trip, say t_{pr} (the notation will be explained shortly), is

$$t_{pr}=2L/c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

proper time

and this will be the time interval as recorded by the clock on the car. A passenger on the car will therefore note that his clock records a time t_{pr} for the journey from start to finish. So far as the clocks set out along the track are concerned the light follows path ABC. The length of this path is, applying Pythagoras' theorem, $2\sqrt{L^2+(l/2)^2}$, where l is the distance travelled by the car between the two events. Since the velocity of light is also c to the ground-based observer (remember that the Michelson-Morley experiment showed the velocity of light to be independent of the velocity of the observer), the time taken for the trip, say t_{im}, which will be the time interval recorded in the photographs of the two separated roadside clocks, is

$$t_{im}=\frac{2\sqrt{L^2+(l/2)^2}}{c}$$

$$=\frac{2}{c}\sqrt{L^2\left[1+\frac{1}{L^2}\left(\frac{l}{2}\right)^2\right]}$$

$$=\frac{2L}{c}\sqrt{1+\left(\frac{l}{2L}\right)^2}$$

So from equation 1

$$t_{im}=t_{pr}\sqrt{1+\left(\frac{l}{2L}\right)^2} \quad \ldots\ldots\ldots \quad (2)$$

improper time

The two timings are different. But how different?

Suppose the experiment is performed in a car with $L=1.5$ m moving at 40 m s^{-1}. Remembering that light travels at 3×10^8 m s^{-1} calculate the two timings.

In this example

$$t_{pr}=\frac{2L}{c}=\frac{2\times1.5\text{ m}}{3\times10^8\text{ m s}^{-1}}$$

$$=1.0\times10^{-8}\text{ s}$$

During this time the car will have moved a distance l given by:*

$$l=\text{velocity}\times\text{time}$$

$$=40\text{ m s}^{-1}\times1.0\times10^{-8}\text{ s}$$

$$=4.0\times10^{-7}\text{ m}$$

Substituting this value of l into equation 2 gives

$$t_{im}=t_{pr}\sqrt{1+(l/2L)^2}$$

$$=\frac{1.0}{10^8}\sqrt{1+\left(\frac{4.0}{10^7\times2\times1.5}\right)^2}\text{ s}$$

$$=\frac{1.0}{10^8}\sqrt{1+\frac{1.77}{10^{14}}}\text{ s}$$

* *Strictly speaking this can only be an approximate calculation, since we are not told whether the clock used in determining the car's speed was mounted on the car or whether roadside clocks were employed. The calculation which is given is only exact if the 40 m s^{-1} was deduced by dividing the distance travelled by the car as measured along the track by the time for the journey as measured by a clock on the car.*

In evaluating the terms under the square root sign, we use the result obtained from the so-called binomial expansion, that when x is small compared to unity

$$(1+x)^m \approx 1+mx+\text{second order terms which may be ignored*} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Here $x=1.77\times 10^{-14}$ and $m=\frac{1}{2}$ so that,

$$t_{\text{lm}}=\frac{1.0}{10^8}\left[1+\tfrac{1}{2}\left(\frac{1.77}{10^{14}}\right)\right] \quad \text{to a very good approximation}$$

i.e. $t_{\text{lm}}=1.0000000000000089\times 10^{-8}$ s

cf. $t_{\text{pr}}=1.0000000000000000\times 10^{-8}$ s

In everyday life the difference between t_{lm} and t_{pr} may therefore be ignored.

Try this problem (Self-Assessment Question 2, *SAQ* 2), which for revision purposes is also included with the others at the back of the Unit.

***SAQ* 2**
Repeat the calculation on the assumption that the car is moving at a speed of 2.0×10^8 m s^{-1}. As before $L=1.5$ m. (This speed again being deduced, using the readings of the clock on the car.)

You will find the detailed working out of self-assessment problems at the end of the Unit, in this case on p. 57. You should follow through the worked solution if you are unsuccessful at arriving at it yourself. You should also read through the working if you obtain the correct answer but are uncertain why.

The answer you should have obtained is:

$t_{\text{lm}}=1.20\times 10^{-8}$ s

$t_{\text{pr}}=1.00\times 10^{-8}$ s

The problem is worked out on page 57.

So, when the speed of an object approaches that of light, the difference between the two timings becomes significant. Indeed, we should, in principle, always specify which time we mean, even with cars. When we use one clock, such as the clock on the car, which is present at both events, we say that we have measured a *proper time interval* (hence our notation t_{pr}). When we have two events occurring at different places, ones which could not be measured by the same clock, we say we have measured an *improper time interval* (hence t_{lm}). Which of these times we care to choose in calculating the car's speed is up to us.

3.3.2 Proper and improper velocities

proper and improper velocity

There are two clear ways of specifying the car's speed. One can either divide the roadside distance l by the journey time t_{pr}, as measured by a clock carried in the car, or can divide l by the journey time t_{lm} measured by the roadside clocks. The first of these calculated velocities is, not surprisingly, referred to as the *proper velocity* (written v_{pr}), while the second alternative is called the *improper velocity* (written v_{lm}). In symbols:

$$v_{\text{pr}}=\frac{l}{t_{\text{pr}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and

$$v_{\text{lm}}=\frac{l}{t_{\text{lm}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Can you say whether the usual car speedometer is calibrated to show proper or improper velocity?

To answer this, we must know whether, in calibrating the master speedometer (against which production speedometers are presumably checked), a clock was carried in the car, or clocks were set out alongside the road.

* *You should satisfy yourself that when used, for example, to evaluate* $(1+0.03)^2$ *the binomial expansion (with* $x=0.03$ *and* $m=2$*) does lead to nearly the same answer as that obtained by squaring* 1.03.

At a guess it seems more likely that roadside stop-watches were employed so that a car speedometer measures improper speeds. However, as has been shown, the difference between t_{pr} and t_{im} and therefore between v_{pr} and v_{im} can be ignored in this case. But, as has also been shown, the difference in timings and therefore in velocities becomes significant when the speed of the object approaches that of light.

Having defined what is meant by velocity, we may rewrite equation 2 which relates proper and improper times, as follows:

$$t_{im}=t_{pr}\sqrt{1+\left(\frac{l}{2L}\right)^2} \quad \ldots\ldots\ldots (2)$$

From equation 5,

$$l=v_{im}\, t_{im},$$

while from equation 1,

$$2L=ct_{pr}.$$

So

$$t_{im}=t_{pr}\sqrt{1+\left(\frac{v_{im}t_{im}}{ct_{pr}}\right)^2}$$

Squaring both sides,

$$t^2_{im}=t^2_{pr}\left(1+\frac{v^2_{im}t^2_{im}}{c^2t^2_{pr}}\right)$$

$$\frac{t^2_{im}}{t^2_{pr}}=1+\frac{v^2_{im}t^2_{im}}{c^2t^2_{pr}}$$

or

$$\frac{t^2_{im}}{t^2_{pr}}\left(1-\frac{v^2_{im}}{c^2}\right)=1$$

therefore

$$t^2_{im}=t^2_{pr}/\left(1-\frac{v^2_{im}}{c^2}\right)$$

so

$$t_{im}=t_{pr}/\sqrt{1-\frac{v^2_{im}}{c^2}}$$

or

$$t_{pr}=t_{im}\sqrt{1-\frac{v^2_{im}}{c^2}} \quad \ldots\ldots\ldots\ldots (6)$$

The clock (notice the singular) on the apparatus, the one most people would consider to be 'moving', measures a smaller interval between the two events than do the other recording clocks (notice the plural). This effect is known as *time dilation*. Here is another example of dilation.

time dilation

SAQ 3

In 1965 Ron Clarke established a world record for the 10 000 metre event of 27 mins. 39.4 secs. Had Clarke carried a watch, how would his timing have differed from the judges? Needless to say the judges were stationary alongside the track.

The problem is worked out on p. 57.

Clarke's timing would have been 0.9999999999999998 of the judges, i.e. it would have been less than theirs by 2 parts in 10^{16}. Of course no present-day clock can measure times to such accuracies. So, unless there is a drastic reduction in the velocity of light (*a priori* there is no reason why the velocity of light should not change over the centuries), no physicist will query which timing is actually employed in athletic events. Should we wish to do so, we may of course express these different timings in terms of proper and improper velocity.

If in our defining relation for v_{pr} (equation 4) we substitute for t_{pr} from equation 6, we obtain

$$
\begin{aligned}
v_{pr} &= l/t_{pr} \\
&= l/t_{im}\sqrt{1-v^2_{im}/c^2} \\
&= v_{im}/\sqrt{1-v^2_{im}/c^2} \quad \ldots\ldots\ldots (7)
\end{aligned}
$$

since v_{im} is defined as

$$v_{im} = l/t_{im} \qquad \text{(equation 5).}$$

Equation 7 emphasizes once again that, unless the (improper) speed of an object approaches the speed of light, the difference between v_{pr} and v_{im} may safely be ignored. We shall make use of equation 7 later in the Unit.

3.3.3 Evidence for time dilation

While time dilation is never likely to be observed with large, slow-moving objects like motor cars, it does appear in a really dramatic way with small, fast-moving particles like muons. Muons break up spontaneously with a half-life of 1.53×10^{-6} s, as measured when they are at rest relative to the observer (stopped in a block of lead, for instance). This is a *proper* time.

Do you remember what half-life means?

If, for example, 1 024 muons are present at any one time, then, on average, 512 will be present after 1.53×10^{-6} (proper) seconds have passed, 256 after $2 \times 1.53 \times 10^{-6}$ s, 128 after $3 \times 1.53 \times 10^{-6}$ s, and so on.

In general, out of an initial number N, the number N_p remaining after a time t_{pr} is

$$N_p = N \times (\tfrac{1}{2})^{t_{pr}/T_{\frac{1}{2}}} \ldots\ldots\ldots\ldots (8)$$

where $T_{\frac{1}{2}} = 1.53 \times 10^{-6}$ s.

This is equation 3 of Unit 2, except that we have indicated that proper times are involved.

Essentially, this experiment with fast muons consists in comparing the number of muons arriving at the top of a mountain with the number arriving at sea level.

Muons arrive with a wide range of velocities and so the apparatus at the top of the mountain is arranged so as to select only those muons with velocities within a narrow range, and the apparatus at sea level is similarly arranged to select only those muons with velocities within the same narrow range. This is done by using an arrangement of lead blocks and scintillation counters like the one you saw in the television programme of Unit 2. (See also the diagram in the broadcast notes on that programme.) With this arrangement, muons that have enough velocity to penetrate the upper lead block but not enough to penetrate the lower block as well will stop in the lower block, decay and be counted. If the upper block is very much thicker than the lower one, the range of velocities selected will be quite narrow. Since the apparatus used to select (or 'measure') the muons' velocity is at rest relative to the muons, the velocity thus measured is an *improper* velocity. (Or is it a *proper* velocity? If you have any doubt go back and read section 3.3.2 again.)

The apparatus on top of the mountain and the one at sea level are arranged to select the same range of velocities—for instance, from $0.991c$ to $0.993c$, or $(0.992 \pm 0.001)c$.

Would the upper lead block have *exactly* the same thickness in the mountain-top apparatus as it has in the sea-level apparatus?

The column of air between sea level and mountain top height would itself add to the stopping power of the upper lead block in the sea-level apparatus. So one would add an *equivalent* thickness of lead to the upper block in the mountain-top apparatus if one wanted the velocities selected at the two levels to be accurately equal.

In a particular experiment, the velocity selected was 0.992c and the height of the mountain above sea level was 1 920 m.

Clearly, to travel down from mountain-top level to sea level, muons of this velocity took a time

$$\begin{aligned} t &= 1\,920/0.992c \text{ seconds} \\ &= 1\,920/0.992 \times 2.998 \times 10^8 \text{ s} \\ &= 6.46 \times 10^{-6} \text{ s.} \end{aligned}$$

Is this a *proper* time or an *improper* time?

It is an improper time, as it is derived from the improper velocity $v_{\mathrm{im}} = 0.992c$. If you have any *doubts* about this, refer back again to section 3.3.2, equation 5.

Thus, $$t_{\mathrm{im}} = 6.46 \times 10^{-6} \text{ s.}$$

How many muons will have decayed in transit between the two levels?

Look again at equation 8. This tells us that if there are N muons initially then the number of muons left after a *proper* time interval t_{pr} will be

$$N_{\mathrm{p}} = N \times (\tfrac{1}{2})^{t_{\mathrm{pr}}/T_{\frac{1}{2}}} \ldots\ldots\ldots\ldots (8)$$

where T is the *proper* half-life time, measured, as t_{pr} is also, in a frame in which the muons are at rest.

To use equation 8 to find the answer to our problem, we have first to convert the improper time we have measured into a proper time. Refer back to equation 6.

$$\begin{aligned} t_{\mathrm{pr}} &= t_{\mathrm{im}}\sqrt{1 - v^2_{\mathrm{im}}/c^2} \ldots\ldots\ldots (6) \\ &= 6.46 \times 10^{-6}\sqrt{1-(0.992)^2} \\ &= 0.815 \times 10^{-6} \text{ s.} \end{aligned}$$

Note that the time, as measured by an observer on the Earth, is nearly eight times longer than the time as measured by the muon.

Now, the number of muons of velocity $v_{\mathrm{im}} = (0.992 \pm 0.001)c$, counted during a certain time-interval by the mountain-top apparatus, will be a given fraction of *all* the muons of that velocity that are passing in that time interval through the atmosphere at that altitude.

Similarly the number of muons of the same velocity, counted by the sea level apparatus during a certain time interval, will be a given fraction of all the muons of that velocity that are arriving at sea level in that time interval. We assume that the two fractions are the same, because the two 'muon sampling' arrangements are the same.

In other words, we assume that

number of muons at S $= k \times N_{\mathrm{S}}$, and that

number of muons at M $= k \times N_{\mathrm{M}}$

where k is the constant fraction of muons (of the given velocity) sampled by the apparatus on the mountain top, M, or by the apparatus at sea level, S (Fig. 3).

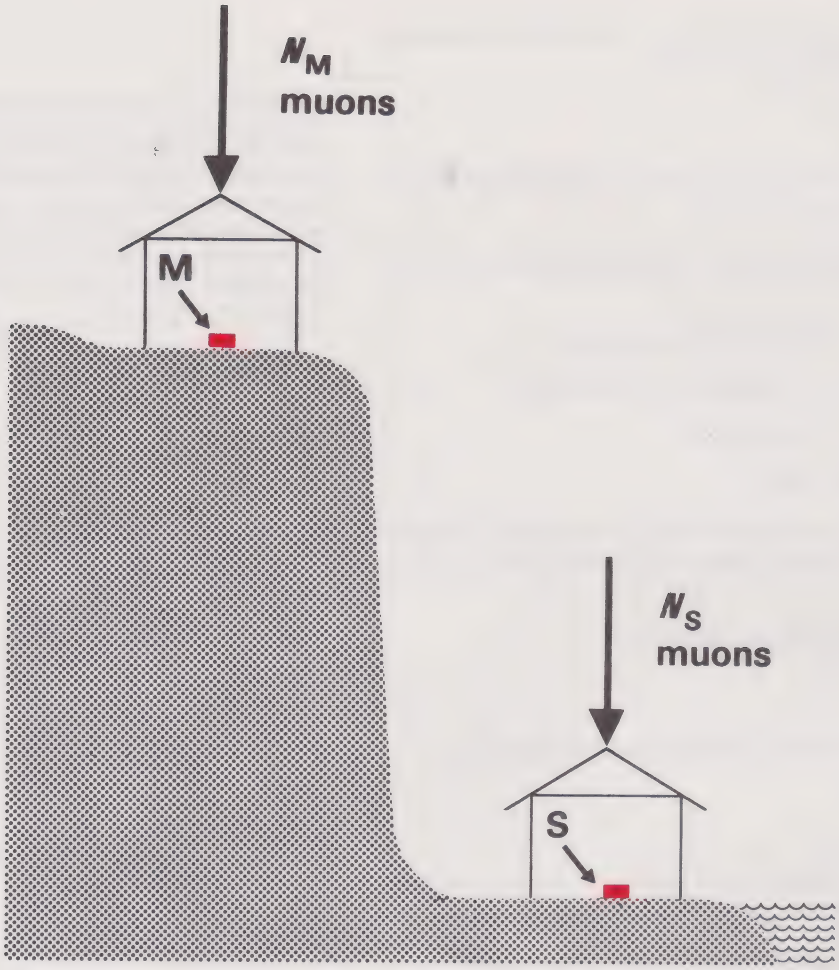


Figure 3 An experiment demonstrating time dilation. The number of muons passing through counter M in a laboratory on top of a mountain is compared with the number passing through counter S in a laboratory at sea level. The experimentally measured ratio for these two counts agrees with the ratio to be expected, only if the muons 'keep' proper time.

So, $$\frac{\text{number of muons at S}}{\text{number of muons at M}}=\frac{N_S}{N_M} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

In the particular experiment quoted, the ratio of counts was found to be

$$\frac{N_S}{N_M}=0.691.$$

Does this agree with the ratio that can be predicted from equation 8? If we use the value of t_{pr} we calculated from equation 6 and the value of $T_{\frac{1}{2}}$ found from the experiment in Unit 2, then the ratio N_s/N_m in equation 8 must be the same as the ratio of muons at S to muons at M.

So

$$\frac{\text{number of muons at S}}{\text{number of muons at M}}=\frac{N_p}{N}=(\tfrac{1}{2})^{t_{pr}/T_{\frac{1}{2}}}$$

$$=(\tfrac{1}{2})^{0.815\times10^{-6}/1.53\times10^{-6}}$$

$$=(\tfrac{1}{2})^{0.533*}$$

$$=0.691.$$

The close agreement between the predicted and measured count ratio shows that the use of equation 6 to convert the measured *improper* time to the proper time was in fact correct.

* *If you are not sure what is meant by* $(\frac{1}{2})^{0.533}$, *you should read section* 1.*A of MAFS. If you cannot evaluate quantities like* $(\frac{1}{2})^{0.533}$, *you should read section* 1.*B of MAFS.*

What result would we have predicted if we had not taken time dilation into account?

Had we used 'common sense' and decided that the flight time 'must' be the same to the muon as it is to the ground based observers, we would have used $t_{\mathrm{lm}} = 6.46 \times 10^{-6}$ s in equation 8. This would have led us to calculate the count ratio as

$$\frac{\text{number of muons at S}}{\text{number of muons at M}} = (\tfrac{1}{2})^{6.46 \times 10^{-6}/1.53 \times 10^{-6}}$$

$$= (\tfrac{1}{2})^{4.21}$$

$$= 0.054.$$

This is some thirteen times less than the observed ratio.

Evidently, this sort of 'common sense' is not always to be trusted!

Another consequence of the difference between proper and improper time is that we must revise our ideas about the *simultaneity* of events that occur at different places. This problem is discussed, with the aid of animations, in the TV programme for this Unit.

You may now attempt *SAQ* 4, p. 54, if you wish.

3.4 Newton's First Law

Having discovered that there is more than meets the eye to the measurement of time intervals, we can examine in more detail the behaviour of large (or *macroscopic*) objects which are moving at speeds significantly less than the speed of light.

How fast can an object move before the discrepancy between proper and improper time reaches 1 per cent?

Substituting $t_{pr}/t_{im}=99/100$ into equation 6 gives

$$\frac{99}{100}=\sqrt{1-\frac{v_{im}^2}{c^2}}$$

On squaring,

$$\left(\frac{99}{100}\right)^2=1-\frac{v_{im}^2}{c^2}$$

Therefore

$$\frac{v_{im}}{c}=\sqrt{1-\left(\frac{99}{100}\right)^2}$$

$$=\sqrt{\frac{100^2-99^2}{100^2}}$$

or, factorizing the numerator*

$$=\sqrt{\frac{(100+99)(100-99)}{100^2}}$$

$$=\sqrt{\frac{199}{100^2}}$$

Therefore

$$v_{im}=\frac{14.1}{100}\,c.$$

So unless the speed of the object exceeds 14 per cent of that of light, i.e. is greater than 4.2×10^7 m s^{-1}, we can safely ignore the difference between proper and improper time, or, if you prefer, the difference between proper and improper velocity. Even with present-day space rockets moving at 10^5 m s^{-1}, it would be pedantic to enquire who held the clocks! But should we wish to speculate about how future space rockets, moving at speeds approaching that of light, might behave, or should we wish to describe the behaviour of nuclear particles, which can readily be made to move at these speeds, then we must make clear which time (or which velocity) we mean.

Here are a few simple experiments to try. Indeed you will probably consider them so simple and so predictable that you may be tempted to 'skip' them. Find something that can roll freely in any direction.

Experiment 1

Try to place a ping-pong ball, marble or whatever on a smooth level surface, so that it stays still. If it rolls away you will no doubt proceed to

* *To show that $x^2-a^2=(x+a)(x-a)$, multiply out the right-hand side. It is $x^2-xa+ax-a^2$, which is indeed x^2-a^2.*

'level' the table. By your actions you are effectively eliminating any sideways pull which the Earth may exert on the object. You may now be tempted to assert that 'every object remains at rest when left alone'—a totally unwarranted generalization.

Experiment 2

If you have access to a record player, place the ball on the stationary turntable. If the turntable has a rubber mat, remove it. If the base is ribbed or the mat is fixed, use a sheet of thin card (several layers of paper will also do) to obtain a flat surface.

What happens?

Should the ball roll the table clearly needs levelling. Now lift the ball off the turntable, switch on the motor, preferably at 16 or 33 revolutions per minute and allow the ball to move in circles, keeping it (by hand) just above a fixed mark on the turntable. Finally, place the ball on the marked spot and let it move freely. In other words repeat experiment 1 but this time from the point of view of someone travelling on the turntable. Should you have no record player you can try the experiment on a flat pad which someone else rotates by hand. An icing stand for a cake is another possibility.

What happens?

Certainly the ball behaves differently in a rapidly rotating frame of reference than on *terra firma*. The words *frame of reference*, or simply *frame*, are a shorthand way of saying 'the laboratory in which the experiment is performed': in this case the laboratory is the turntable. More formally, the frame is the set of co-ordinate axes in which the behaviour of the object is represented graphically.

frame of reference

Try and deduce how the ball would appear to move to an observer standing on the turntable, i.e. how it would move in the frame of reference of the turntable. One technique you might experiment with is to dip the ball in ink before placing it on the turntable, which has been covered with a piece of graph paper. The graph showing the motion of the ball as seen in the turntable frame will be drawn automatically. Whatever technique you adopt you will probably conclude that 'in the frame of reference of the turntable a body left alone on a flat surface accelerates outwards along a radius'. Our cautionary words about over-generalizing on the basis of a single experiment (experiment 1) were clearly justified!

Experiment 3

Repeat experiments 1 and 2, this time setting the ball going at constant speed. As observed in one of these frames, but only in one of them, the ball continues to move at constant speed in a straight line.

Because of the experiments you carried out in Unit 1 you may well be suspicious of any experiment carried out with a ball. The conclusions are however quite general so long as essentially frictionless conditions prevail. The experiments may, for example, be repeated with the same results using small laboratory 'hovercraft' or 'pucks'—flat discs kept hovering by means of an air blast. In one type of apparatus, air streams through a multitude of small holes in a flat table and the discs ride on the air cushion. This is, of course, an inversion of the usual hovercraft principle. You will see such an apparatus in operation in this Unit's TV programme. Once set going in a 'fixed' laboratory (one on the Earth), a puck continues to move at constant speed in a straight line. This may best be demonstrated by

taking a time-exposure photograph while the table is illuminated by a stroboscope (a device which produces short bursts of light at fixed intervals). Figure 4 shows such a photograph. The successive images are, as you can quickly discover, equally spaced in a straight line.

Figure 4 A stroboscopically lit photograph showing a puck moving freely at constant speed along a horizontal air table. In taking this type of picture the camera shutter is left open; the successive images show where the puck was at each flash from the stroboscope. The time interval between flashes is constant.

We may summarize our experiments by saying that there are some frames of reference where the law that 'a body remains at rest or continues to move in a straight line at constant speed when left to itself' holds true. This law is known as *Newton's First Law of Motion*, and frames where it is true are known as *inertial frames of reference*.

Newton's First Law

Not surprisingly, frames where the law is not true are called *non-inertial*.

inertial and non-inertial frames of reference

In which of the following 'laboratories' will Newton's First Law hold true?

(a) In a train moving at constant speed in a straight line.
(b) In a car turning a tight circle at constant speed.
(c) In a decelerating bus moving along a straight road.

An appeal to everyday experiences, such as the knowledge of what happens if you don't hold on when the bus brakes, should convince you that only (a) is inertial.

Just as optical experiments (such as for example the Michelson-Morley experiment) failed to show whether or not the laboratory was moving (at constant speed in a straight line), so have our simple mechanical tests. Newton's First Law holds true in a train 'moving' at a steady 0 m s^{-1} (at rest), or at a steady 50 m s^{-1}. Those TV pictures of floating toothbrushes in the Apollo spacecraft are proof enough that Newton's First Law also holds in a spaceship moving at a steady 10^5 m s^{-1}.

Similarly, no biochemical test reveals uniform linear motion. A cup of tea in a station buffet tastes the same as does one made to the same recipe on an express. On the other hand a cup of tea in a violently swaying carriage, which is therefore accelerating and decelerating, seems decidedly uninviting—our stomach knows when we are changing speed.

Our next group of experiments is going to be carried out in an inertial frame. Perhaps the most readily available inertial frame is a horizontal table on the Earth. However because the Earth rotates such a frame is strictly speaking non-inertial, but for most purposes the discrepancy is small. As you will learn in Unit 22, the discrepancy must sometimes be taken into account. In the vertical direction the Earth is not even approximately inertial; witness what happens when you release an object held above the Earth's surface.

You may now attempt *SAQ* 5, p. 54, if you wish.

3.5 Newton's Second Law

3.5.1 Arbitrary units

It is common knowledge that to change the speed of a motor car one must step either on the brakes or on the accelerator; pushes or pulls are needed to accelerate an object. We are now going to make a systematic study of just how the motion of a body changes under the influence of a force.

The first problem is how to get a force, a reproducible force. You have probably tried stretching a spring, at some time or other, and will have got the subjective impression that it always takes the same effort to keep any one spring stretched the same amount.

Let us therefore agree, albeit tentatively, that when a particular spring is stretched a defined amount it provides a fixed reproducible force. It is up to us to choose the spring and to say how far it must be stretched to provide the unit of force. For want of a better name let us call this unit of force a *strang*; you will not find this unit listed in any table of units. It is a new unit we are temporarily adopting. Later we shall abandon the strang in favour of a unit based on a more fundamental definition of force.

strang

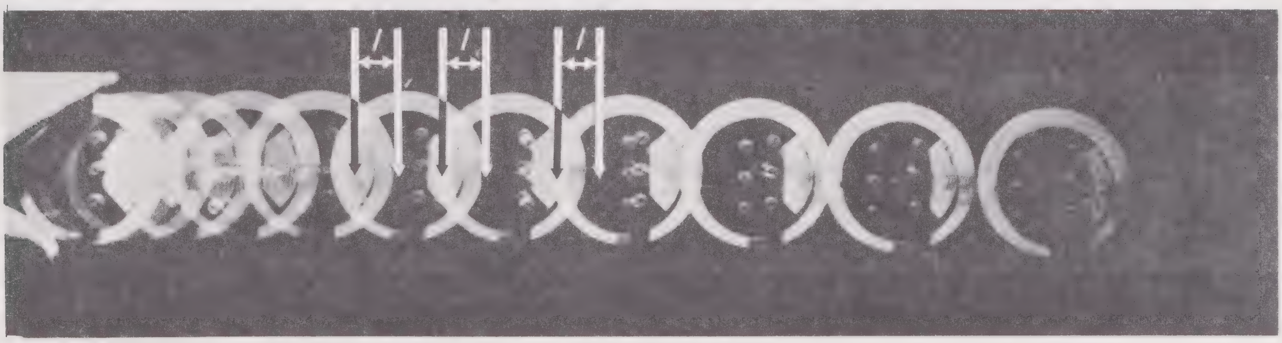


Figure 5 (a)

Figure 5 The effect of applying a force of 1 strang to 1 puck. (a) A stroboscopically lit photograph of the experiment. The arrowed lines indicate the length of the spring. (b) A schematic diagram of the apparatus used to obtain Figure 5a.

To investigate what a force of one strang does to a puck, the spring is attached to such a puck which is then pulled so as to stretch the spring by the agreed amount. On releasing the puck it speeds away, as expected. Figure 5a shows the result of such an experiment carried out under stroboscopic illumination. You should measure up the spring length to satisfy yourself that a constant force has been applied throughout the experiment. You should also satisfy yourself that the puck accelerates in the direction of the applied force. To study the acceleration in more detail, whether for example it is constant or not, we should plot a graph of . . .

What should we plot?

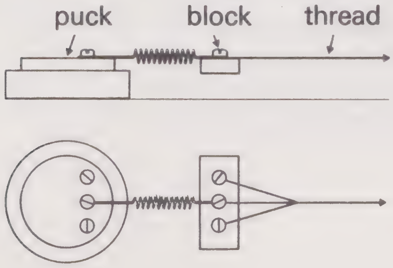


Figure 5 (b)

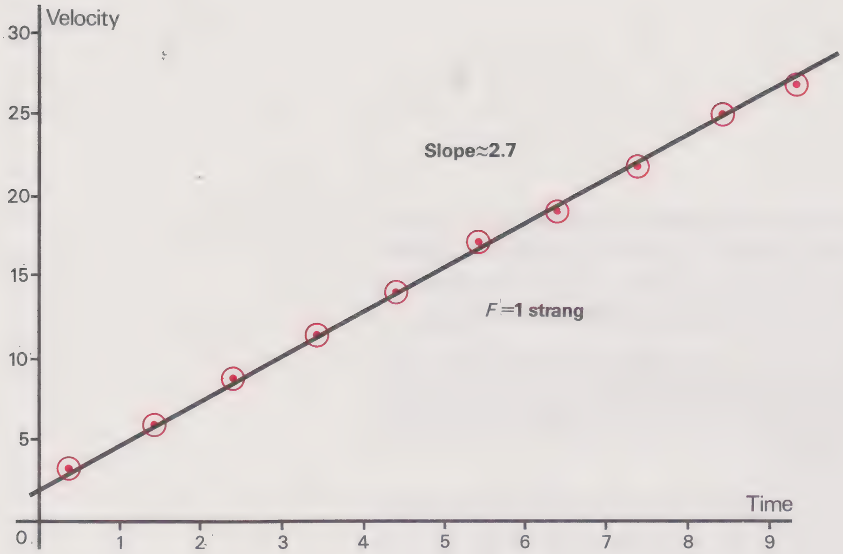


Figure 5 (c)
A graph of velocity of the puck against the time when it had this velocity. The velocity is obtained from the distance between successive images; the puck is assumed to have this velocity at a time midway between the times at which the two neighbouring images were recorded.

If you did not think of plotting a graph of velocity against time or one of distance against $(\text{time})^2$ then you should now read Appendix 1. Figure 5c shows the result of making the first of these plots. From this plot you will be able to deduce that, when acted upon by a given force, the acceleration of the puck (the rate of change of its velocity) is constant, with a value given by the slope of the graph.

The following examples, detailed working of which are to be found on pp. 58–59, will allow you to check on your understanding of elementary kinetics, i.e. of the terms acceleration, velocity, distance and time, and their interrelations.

These questions assume you are familiar with the contents of Appendix 1 (Red).

SAQ 6
A car accelerates away from rest at 4 m s^{-2}. What is its velocity after it has gone 8 m?

Answer: 8 m s^{-1}. The problem is worked out on p. 58.

SAQ 7
A sprinter who is travelling at 2 m s^{-1} accelerates at a constant rate while he travels 8 m. His final velocity is 5 m s^{-1}. How long did he spend accelerating?

Answer: 2.3 s. The problem is worked out on p. 58.

SAQ 8
A puck starting from rest accelerates at a constant rate for 3 s through a distance of 5 m. What is the puck's acceleration?

Answer: 1.1 m s^{-2}. The problem is worked out on p. 59.

The next experiment we might try with the puck is to investigate the way the acceleration depends on the accelerating force. But this raises the problem of how to vary the force. Again we must resort to everyday experience—namely that it is twice as hard to keep two springs stretched when side by side as it is to keep one stretched the same amount. So we agree that if we take two parallel springs identical with the one we employed in the first investigation and stretch both the agreed amount we provide a force of two strang. With three springs the force is three strang, etc. We have now defined how forces may be added.

How do we ensure that the springs are identical?

Figure 6 (*a*)

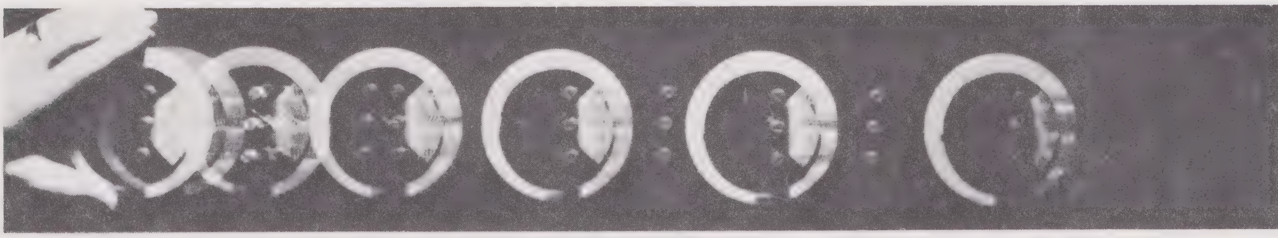


Figure 6 (*b*)

Figure 6 The effect of applying different forces to a body of constant mass. (a) A stroboscopically lit photograph showing the effect of applying 2 strang to 1 puck. (b) The effect of 3 strang on 1 puck. (c) The corresponding graphs of velocity against time.

We should repeat the first experiment and check that all springs give the same result. Figure 6a shows the effect of applying 2 strang to one puck and Figure 6b shows the effect of 3 strang. The appropriate velocity versus time graphs are shown in Figure 6c. These plots demonstrate that with a force, F, of 2 strang the acceleration, a, is double its 1 strang value; that when F is 3 strang, a is trebled; and that, in general, the acceleration of a given body is proportional to the accelerating force:

i.e. $$a \propto F \qquad (10)$$

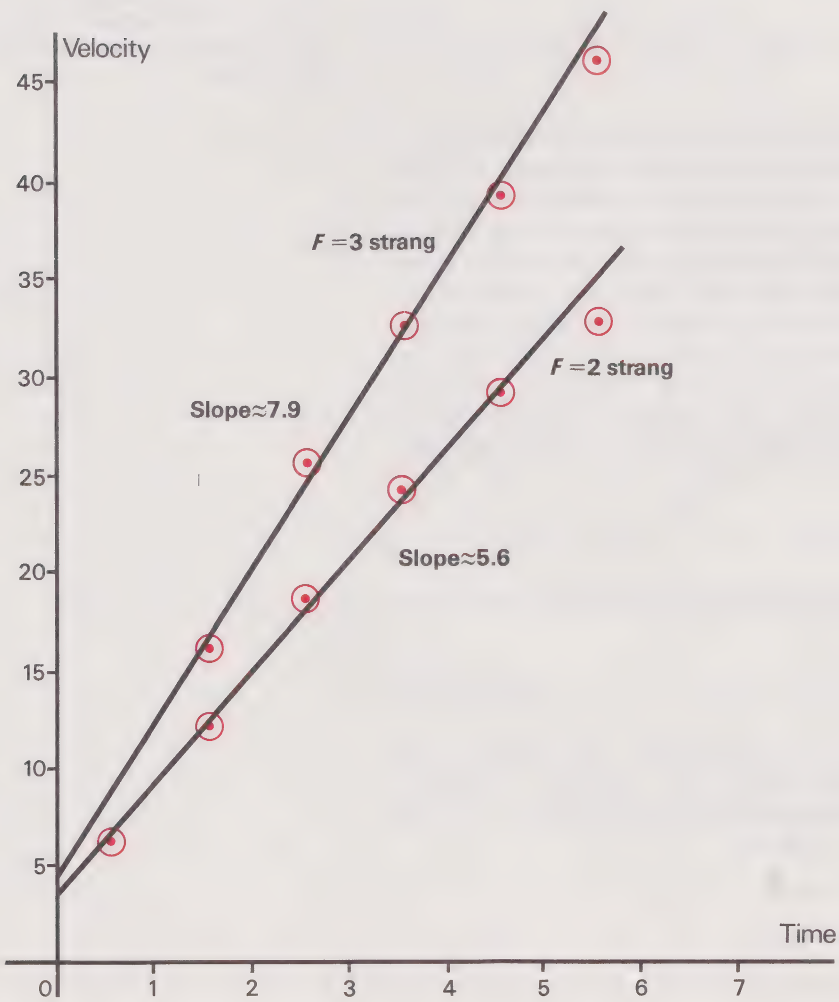


Figure 6 (*c*)

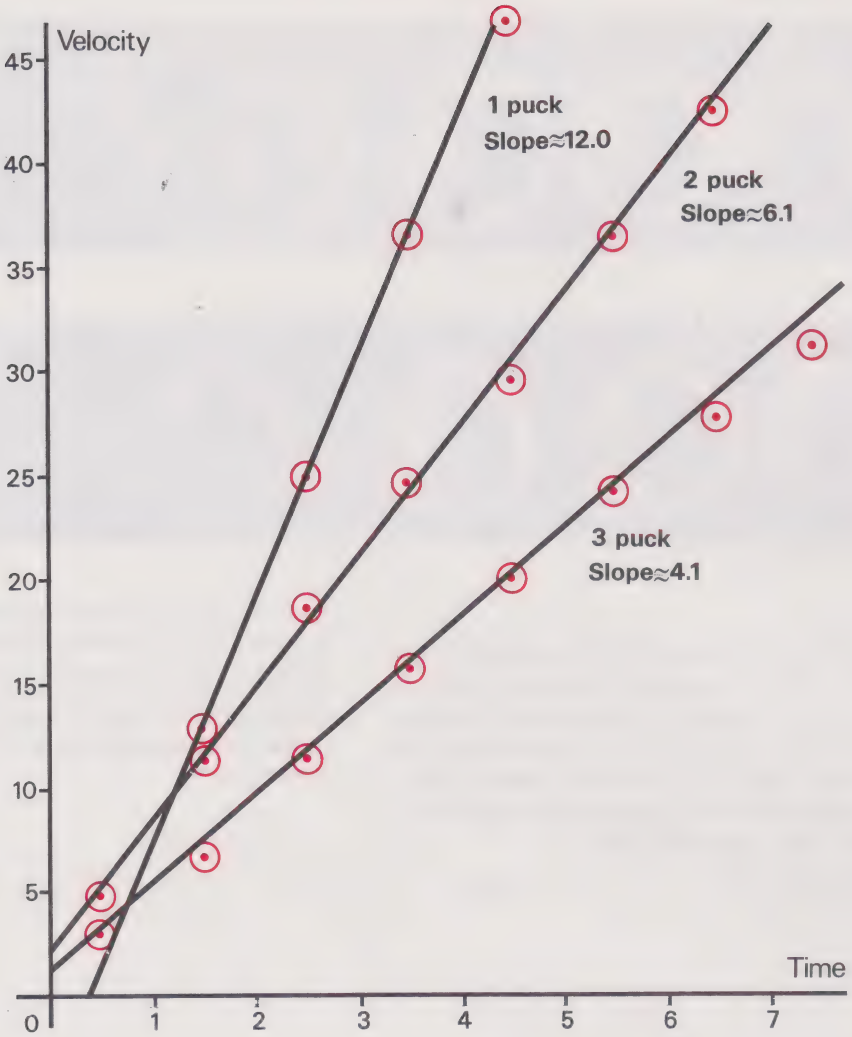


Figure 7 Shows graphically, by means of a plot of velocity against time, the effect of applying a constant force to 1, 2 and 3 pucks.

The next step is to vary the amount of material being accelerated (i.e. the mass, m) while keeping the accelerating force constant. Again we must agree on how to vary the mass in a controlled way. It seems obvious that the mass can be doubled by stacking two identical pucks on top of each other; trebled by stacking three identical ones, etc. (by identical we mean that each behaves in the same way when used separately in our earlier experiment). Figure 7 shows graphically the effect of applying a constant force to 1, 2 and 3 pucks. On doubling the mass, the acceleration, as measured by the slope of the graph, halves; on trebling the mass the acceleration is but a third of its one-puck value. In general, the acceleration produced by a constant force is inversely proportional to the accelerated mass.

$$a \propto \frac{1}{m} \quad \text{....................(11)}$$

The results of our two sets of experiments may be combined in the single relation

$$a \propto \frac{F}{m} \text{.....................(12)}$$

To satisfy yourself that equation 12 does combine the results of the separation investigations, notice that if you keep F constant in equation 12 then equation 11 is retrieved; keep m constant and equation 10 is the result. Cross-multiplying equation 12 gives

$$F \propto ma$$

or

$$F = Kma \text{ ....(13)}$$

where K, the constant of proportionality, is determined by experiment.

This general relation is known as *Newton's Second Law*. In one such experiment a force of 2 strang produced an acceleration of 1.38 m s^{-2} in a mass of 1 puck.

Newton's Second Law

$$\therefore \quad K = \frac{F}{ma} = \frac{2}{1 \times 1.38} \quad \frac{\text{strang}}{\text{puck m s}^{-2}}$$

$$= 1.45 \text{ strang s}^2 \text{ puck}^{-1} \text{ m}^{-1}$$

The relation between force, mass and acceleration is therefore

$$F = 1.45\, ma$$

when F is measured in strangs (our strangs), m in pucks (our pucks), and a in m s^{-2}.

You decide to measure the mass of a pineapple by attaching it to a puck. A force of 3 strang gives the combined system an acceleration of 1.27 m s^{-2}. What is the mass of the pineapple?

It may not be very exciting to know that the combined mass of this pineapple and puck is $F/1.45a = 1.63$ puck, i.e. the mass of the pineapple is 0.63 puck, but this calculation should underline the fact that mass determinations are relative determinations.

If other workers reported the relation $F = 7.8\, ma$, would you be surprised?

Not in the least. The relation depends on what springs and pucks are used.

3.5.2 The SI units of mass and force

While there is nothing wrong with basing forces on arbitrary springs, and masses upon arbitrary pucks, it is neither a reproducible nor a permanent system. It is hard to manufacture springs with identical characteristics. In addition, these characteristics will change with age and even with the temperature of the room. A less temperamental system of units is called for. You may recall from Unit 2 how such early and variable units of length as the cubit were replaced by ever more reliable standards over the years and how today's definition of unit length is given in terms of wavelengths of a line in the spectrum of krypton. Likewise the rather variable definition of the second, in terms of the solar system, has been replaced by the definition in terms of a characteristic frequency emitted by the caesium atom. We are now going to introduce the standard of mass. From this standard of mass and those of length and time will emerge the new definition of the unit of force.

There is in a building near Paris an arbitrary lump of platinum known as The Kilogramme (kg). Originally constructed to have a mass closely equal to that of 10^3 cm^3 (10^{-3} m^3) of water at 4° C this particular lump of platinum now stands as the unit of mass in the SI system.

In imagination let us borrow The Kilogramme and place it on a frictionless surface—in practice we would of course have to use a duplicate. Next we attach an uncalibrated spring balance* (one with an unmarked scale) to The Kilogramme and pull with various steady forces until we achieve an acceleration of 1 m s^{-2}. To be exact, we should of course measure distances with some optical device using a line in the spectrum of krypton 86, and measure times (improper times in fact) with a caesium clock. Having achieved an acceleration of 1 m s^{-2}, we agree to call the arbitrary constant

The Kilogramme

* *The type of pocket balance used for weighing luggage might be convenient.*

force that produced this acceleration 1 kg m s^{-2}. Notice how this unit of force is defined by means of a 'thought experiment'—it only requires the kilogram, the metre and the second. There is no need for a 'standard' balance providing '1 kg m s^{-2}'. Indeed, to undertake the *practical* investigation we had to introduce another arbitrary but practical force unit—the strang. To see what value the constant of proportionality between force mass and acceleration will have in our new system of units look again at equation 13.

Deduce *K* in the SI system.

Not only has K the value of unity, but it is dimensionless:

$$K=\frac{F}{ma}=\frac{1 \text{ kg m s}^{-2}}{1\times 1 \text{ kg m s}^{-2}}$$

Provided we are in the appropriate system, which in this case means measuring forces in kg m s^{-2}, masses in kg, and acceleration in m s^{-2},

$$F=ma \quad \dots\dots\dots\dots\dots\dots(14)$$

As it stands this equation does not do full justice to the results of our experiment. The force, besides having a magnitude F, was applied in a definite direction. The acceleration, besides having a magnitude a, was also directed in a specific direction—the direction in which the force was applied. So the force and the acceleration should be given vectorial* representations, i.e. $\boldsymbol{F}$ and $\boldsymbol{a}$ respectively. Equation 14 should properly be written:

$$\boldsymbol{F}=m\boldsymbol{a} \quad \dots\dots\dots\dots\dots\dots(15)$$

We may wish to calibrate an everyday spring balance in units of force. This is easily done: we simply pull a replica Kilogramme until the acceleration is say 5 m s^{-2}, when the force will be 5 kg m s^{-2}. The spot on the scale opposite the pointer can then be marked 5 kg m s^{-2}. When the acceleration is 3 m s^{-2}, the force is 3 kg m s^{-2}, etc. Since it is rather a mouthful to have to keep on saying 'kilogramme metre per second squared' as the unit of force, this is conventionally shortened to the word *newton* (written N). It is only a shortening; whenever the word 'newton' occurs as a unit of force it can be replaced by 'kg m s^{-2}'. Such a balance, calibrated to show the force it can provide in newtons, is referred to as a *newton balance*.

the newton

newton balance

There is little virtue in being able mechanically to substitute numbers into formulae to arrive at the correct solution, but a couple of examples will not be out of place if they can give you a feel for various forces you encounter in everyday life.

What push is required to give a 0.5 kg bag of sugar (i.e. one of about 1 lb.) a constant acceleration of 3 m s^{-2}?

Substituting $m=0.5$ kg and $a=3$ m s^{-2} into equation 14 gives the force required as 1.5 kg m s^{-2}, i.e. 1.5 N. It may be of some interest to ask whether an acceleration of 3 m s^{-2} would be a reasonable acceleration in practice. Suppose you did manage to heave the bag of sugar with this acceleration through a distance of 1 m (about arm's length)—would the final velocity seem plausible? You should be able to convince yourself that the final velocity is $\sqrt{2\times 3\times 1}\approx 2.5$ m s^{-1}. (If you cannot reach this answer for yourself, re-read Appendix 1 (Red).) So the problem was a realistic one.

* *See* MAFS, *section 4.D.*

Here is another problem. This time you will have to make inspired guesses of the quantities.

SAQ 9
Make an order of magnitude estimate of the push that human legs can provide while accelerating away from rest.

The problem is worked out on p. 59.

Now a more formal problem:

SAQ 10
A car of mass 400 kg accelerates away at a constant rate from rest. In 15 s it has reached a speed of 50 km per hour. What force is the engine providing?

Answer: 370 N. The problem is worked out on p. 59.

3.5.3 Vectors

So far, in all the experiments where we deliberately set out to change the speed of an object, we have only been concerned essentially with one force. It is true that we have prescribed how parallel forces are to be added but have said nothing at all about what happens when a body is being simultaneously acted upon by several non-parallel forces. What might happen if, for example, as shown in Figure 8a, a 6 kg puck was simultaneously pulled with a force $F_1 = 5$ N in one direction, and with a force $F_2 = 8$ N at an angle of 70° to this direction (both forces in the plane of the table)?

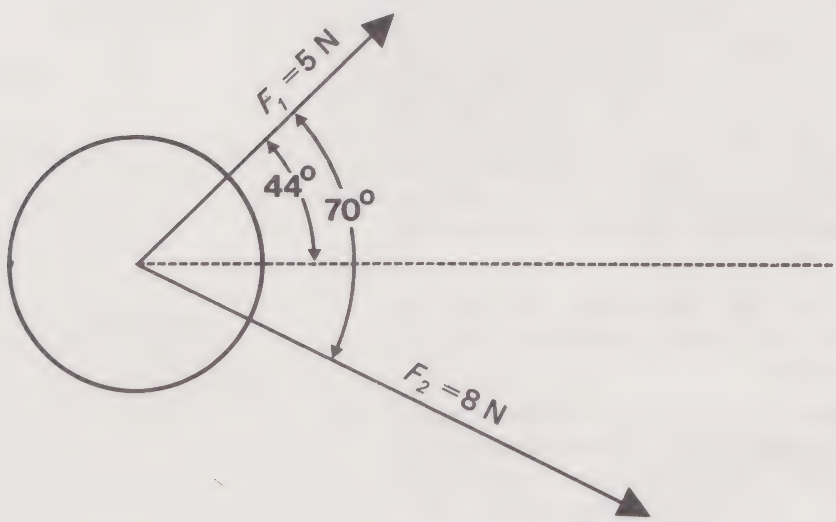


Figure 8 (a) The effect of simultaneously applying two forces to a single body. On applying a force $F_1 = 5$ *N and a force* $F_2 = 8$ *N in the directions indicated, the puck moved along the path shown by the dashed line.*

We know from our everyday experiences that an object acted upon in this manner will accelerate away in some direction between the two forces. To find out exactly what happens, all we need do is to provide two people with balances calibrated in newtons and instruct them to pull on a 6 kg puck with the specified forces in the specified directions. When this particular experiment is performed it is found that the puck accelerates away at 1.8 m s^{-2} along a linear path which lies at 44° to the 5 N force—this path is shown dashed in Figure 8(a). Now think about how someone who was unable to see the springs would describe the experiment.

Such a person would, of course, just see a 6 kg puck accelerating at 1.8 m s^{-2}. But what, quantitatively, would he infer?

Trusting equations 14 and 15, he would deduce that the puck was being acted upon by a force F of 6×1.8 kg m s^{-2}, i.e. 10.8 N, along the direction in which it accelerates. To put it differently, he could reproduce the experimental result for himself by applying a force of 10.8 N in the observed direction of motion. We, however, know that the puck is actually being simultaneously pulled by two forces. Is there any way of combining, on paper, our two pulls of 5 N and 8 N to produce a net force of 10.8 N?

Look at section 4.D on vectors and how they are added, in *MAFS*. Then try to prove that the resultant of two forces of 8 N and 5 N acting at an angle of 70° to each other is indeed a force of 10.8 N acting at an angle of 44° to the 5 N force.

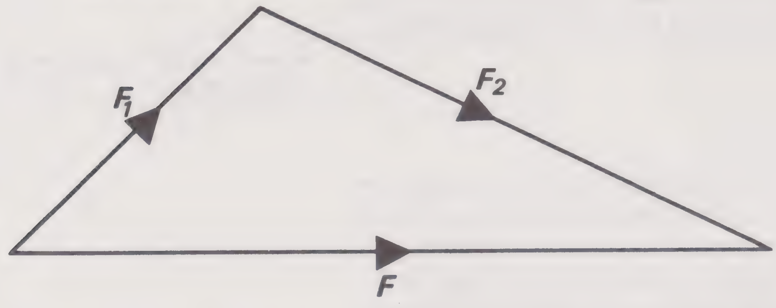


Figure 8 (b) Shows how F_1 *and* F_2 *are added according to the 'head-to-tail' rule to produce a resultant force* F *which does indeed act along the direction in which the puck moves.*

Perhaps the easiest way to add vectors is to use the 'head-to-tail' rule, as shown in Figure 8b. First you represent one of the forces, say, as a vector, i.e. draw a line of length proportional to the magnitude of the force (on some chosen scale) and in the direction of the force. Place an arrow on this line showing the sense in which the force (F_1) acts. Next draw a similar line for F_2, placing the tail of F_2 at the head of F_1. Then join the free head and tail to get a line which represents the resultant force F. You can check for yourself that this line drawn for F does give a magnitude of 10.8 N and is in a direction of 44° to the 5 N force. You can also try drawing F_1 and F_2 in the reverse order. You will get the same result. You may prefer to calculate the resultant by applying trigonometry; the cosine rule will give the magnitude of F and the sine rule its direction. (If you are unfamiliar with the sine and cosine rules see *MAFS*, section 4 D.7.)

Although the forces appear to satisfy the 'head-to-tail' rule, this rule must be treated with caution. We shall shortly see that this rule for adding vector quantities can break down.

Velocity, e.g. the velocity of a motor car, is properly a vector quantity it has direction and magnitude. The magnitude is customarily called the *speed*—although we shall be fairly colloquial in the way we use the terms velocity and speed, frequently using one when we really mean the other. In most situations, the head-to-tail rule makes the correct predictions when used in calculating the resultant velocity of an object, as in the following example.

***SAQ* 11**
An ascending escalator set at 45° to floor level is moving at 0.8 m s^{-1}. Someone is walking up the escalator at 1.1 m s^{-1}. Use the head-to-tail rule to deduce his speed and direction of movement relative to someone stationary at the foot of the escalator.

The problem is worked out on p. 60.

The predicted resultant velocity is 1.9 m s^{-1} in a direction at 45° to floor level; quite in accord with experience. But here is an example where the

'head-to-tail' rule breaks down when used in calculating the resultant velocity of an object.

A π° meson moving at 0.9998 c disintegrates, emitting two photons. Since photons travel at a speed of c it might be argued that the resultant velocity of a photon formed when a π° meson disintegrates could be deduced by applying the head-to-tail rule, in the same way as it was applied in Q.11, namely by the velocity of the photon (of the walker) to that of the meson (of the escalator). Try applying the rule to deduce the final velocity of a photon so formed, assuming it moves in the same direction as did the meson.

The predicted value is 1.9998c.

But what is the observed speed of a photon emitted in such an experiment?

If you have forgotten look back at p. 12. The simple rule quite clearly breaks down when the speeds approach that of light.

3.5.4 Weight

Here is what may look like a trivial problem, but it is one that introduces a new term into your vocabulary.

***SAQ* 12**
What, roughly, is the force of attraction between the Earth and a 'quarter pound' slab of chocolate? You should know that, when released, a slab falls with a constant acceleration. In an actual experiment a slab, in falling from rest through a distance of 3 m, acquired a final speed of 7.5 m s^{-1}. The problem is worked out on p. 60.

The answer is a useful one to remember; about 1 N. The pull of the Earth on an average apple is also about 1 N. Even when the slab is in our hands and is not accelerating, the downward force of attraction between it and the Earth is presumably the same. This force we call the *weight* of the object. To prevent an object accelerating downwards we must provide an upwards force equal to the weight of the object so that the resultant force is nil. The *mass* of an object is the same whether that object is on the Earth or on the Moon; the *weight* of an object is not constant. This basic difference between mass and weight is further emphasized by the different units for the two quantities: mass is measured in kilogrammes (which is a scalar* quantity), and weight in newtons (which is a vector quantity). Unfortunately for us the terms mass and weight are frequently confused in everyday speech. **weight**

An astronaut stationed on the Moon wishes to make sure that his daily sugar consumption is the same as it was on Earth. Should he measure out the same mass, or the same weight of sugar as he did on Earth?

As the astronaut clearly wishes to consume the same volume of sugar as on Earth he must take the same mass. This proportionality between the mass and the volume of a given material is actually a consequence of our definition of how masses are to be added. In section 3.5.1, for example, we took the combined mass of two identical pucks, stacked one on top of the other, to be two pucks; the general form of Newton's Second Law

* *Whereas a* vector *quantity has both magnitude and direction, a* scalar *quantity has magnitude only.*

(equation 14) was arrived at by making this assumption. Thus when we treble the volume of a given material, we are also trebling the mass. Put differently, we are assuming that the ratio of mass/volume of a given material is constant. This ratio, as you probably know, is called the *density* of the material.

density

Should the astronaut 'weigh' out the sugar, perhaps by adjusting the amount hung from the end of a spring balance until the pointer reads the same as it should on Earth, he would end up with about six times as much sugar as he required. The pull of the Moon on any one object, i.e. the weight of that object on the Moon, is only about a sixth of the weight of that object on Earth.

Properly speaking, the mass of objects can only be determined by applying a known force, F, to the object (e.g. in newton) and by measuring the resulting acceleration (e.g. in m s^{-2}). In appropriate units, as in the SI system, $m = F/a$ (equation 14). It is however possible in practice to compare two masses by 'weighing' them on a beam-balance like the one shown schematically in Figure 9. It is an experimental fact that to keep the beam of such an instrument horizontal we must apply forces F_1 and F_2 such that

$$F_1 l_1 = F_2 l_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

where l_1 and l_2 are the corresponding arm lengths.

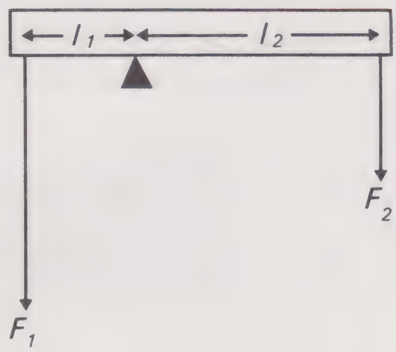


Figure 9 A schematic diagram of a beam-balance. The beam will be horizontal when the two forces F_1 *and* F_2 *are such that* $F_1 l_1 = F_2 l_2$.

In the balance the forces come from the 'weights', i.e. the force of gravitational attraction on the masses m_1 and m_2. Had we dropped the masses m_1 and m_2, they would have had local downward accelerations of, say, g_1 and g_2; so the forces acting on them are $m_1 g_1$ and $m_2 g_2$ respectively (from equation 15). Substituting these forces into equation 16 shows that, when the beam is horizontal,

$$m_1 g_1 l_1 = m_2 g_2 l_2 \quad \ldots\ldots\ldots\ldots\ldots (17)$$

One of the really startling facts of life is that, once air resistance has been eliminated, all objects at any one spot on the Earth fall with the same acceleration, i.e. $g_1 = g_2$. This is a totally unexpected result and should thrill us more than it usually does! Knowing this result we can simplify equation 17 to

$$m_1 l_1 = m_2 l_2$$

With the traditional chemical balance, and indeed the sort of letter balance frequently seen in English village post offices, the two arms are of equal length which simplifies mass comparisons still further. It perhaps should be stressed again that such balances really measure weights; only because of the fact that all bodies fall with the same local acceleration can they be used to compare masses. Of course, in outer space, away from gravitational fields, masses can only be determined by applying a force and measuring an acceleration.

Although we shall, generally, adhere to the SI system of units, you will find us occasionally using units like pound and ton. Being units of mass these should never be used to denote a weight, although this may be excused if the word weight is attached as, e.g., in pound weight (about 4 N).

3.5.5 Centripetal and centrifugal forces

We have now reached the stage when we can discuss the experiment of the ball placed on the turntable, the ball which spirals outwards or, according to someone travelling on the turntable, accelerates outwards in a radial direction. (Experiment 2 of section 3.4.)

Since it is easier to discuss the behaviour of an object which is travelling around in a circle than to discuss that of an object which is spiralling outwards, let us simplify the problem by constraining the object to follow

a circular orbit. You need only recall childhood experience of twirling objects around on the end of a string to realize that to prevent the object from flying outwards you must exert an inward force with your hand.*

This force is called the *centripetal* force, and is denoted by F_i in Figure 10a. We are now going to derive a relation between F_i, the speed v of the object of mass m, and r, the radius of its orbit.

centripetal force

Before reading on, try and decide how F_i depends, in a qualitative way, on m, v, and r. For example, does F_i increase as m increases? It may help to recall childhood experiments to obtain the answers although you may prefer to devise fresh experiments. You may like to make a note of what you decide and later compare it with the final deductions.

Yet another way of arriving at a possible expression for F_i is by means of dimensional analysis. If you have read section 4 on 'Dimensions' in *HED*, you might like to apply this technique to deduce the relation between F_i, m, v and r. This derivation is outlined in Appendix 2 (Black).

We shall now derive an expression for F_i using what you have already learnt of Newtonian mechanics. To someone stationary on the ground the object traversing the circular path will appear at one time, say t, to be moving at a velocity $\boldsymbol{v}_p$. This velocity $\boldsymbol{v}_p$ is shown vectorially in Figure 10a, where the tangent to the circle at P gives the instantaneous direction of movement of the mass as it passes P, while the length of the vector gives the magnitude of the velocity, i.e. the speed. At a later instant of time, which we may denote by $(t+\delta t)$,** the mass has moved to position Q, where we have indicated velocity by $\boldsymbol{v}_q$. Although the *speed* of m is *unaltered*, the direction of movement has changed; in other words the *velocity* of m has *changed*. To find the change in velocity we merely have to discover what velocity must be added to the velocity at P to produce the value at Q.

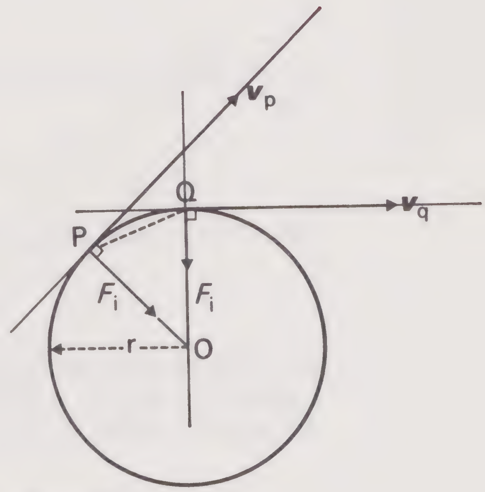


Figure 10 (*a*)

How would you evaluate the change in velocity of m as it moves from P to Q?

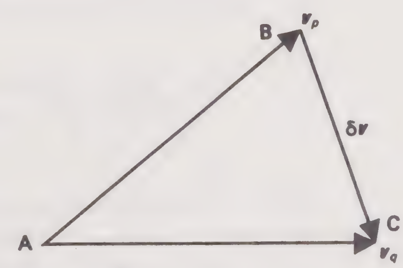


Figure 10 (*b*)

Figure 10 *Centripetal forces.* (*a*) *In a time interval* δt *a body of mass* m *has moved from position P to position Q. During this time interval the* velocity *of the mass has changed from* v_p *to* v_q.
(*b*) *Shows how the change in velocity* δv *of the puck in moving from P to Q is obtained by applying the 'head-to-tail' rule.*

The rules of vector addition must be applied. This has been done in Figure 10b by means of the head-to-tail rule. Here $\delta\boldsymbol{v}$ denotes the velocity which must be added to the velocity $\boldsymbol{v}_p$ at P to produce the velocity $\boldsymbol{v}_q$ at Q. So far as the direction of $\delta\boldsymbol{v}$ is concerned, it makes equal angles with the velocities at P and Q (in triangle ABC, AB = AC). As the time interval δt gets smaller, so $\delta\boldsymbol{v}$ gets smaller, and the angle between the direction of $\delta\boldsymbol{v}$ and the tangent lines to the circle approaches 90°. Thus in the limiting case when δt approaches zero, $\delta\boldsymbol{v}$ is directed inwards along a radius. To find the magnitude of $\delta\boldsymbol{v}$ one need simply recognize that triangles OPQ and ABC are similar; similar because $P\hat{O}Q = B\hat{A}C$, $OP = OQ$, and $AB = AC$.

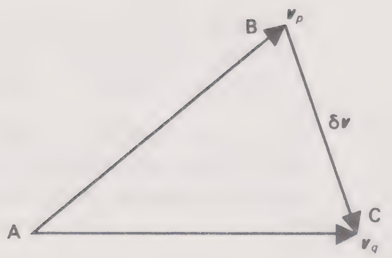
If you cannot remember the properties of similar triangles you should read *MAFS*, section 2.E.

* *Although the experimenter will feel an outwards force directed on him, on his hands for example, the force to consider in acceleration experiments is the force on the object which is being accelerated. When we accelerated pucks we* only *considered the force acting on the puck; we ignored the force which the spring exerted on the puller.*

** *By* δt *we mean a small interval of time.*

From the properties of similar triangles

$$\frac{BC}{AB}=\frac{PQ}{OP} \quad \dots\dots\dots\dots\dots\dots(18)$$



Now the length BC is the magnitude* of the velocity change, δv, the length AB is the magnitude* of the velocity v, and the length OP is equal to the orbit radius r. Note also that as Q approaches P, the length of the chord PQ approaches more and more closely that of the arc PQ. But the arc PQ is the path followed by the mass, moving with velocity v, in the time δt. So its length is $v\delta t$.

If we now substitute BC $=\delta v$, AB $=v$, OP $=r$ and PQ $\approx v\delta t$ in equation 18, we obtain

$$\frac{\delta v}{v}=\frac{v\delta t}{r}$$

So the magnitude of the change in velocity, δv, which occurs in the time δt is

$$\frac{\delta v}{\delta t}=\frac{v^2}{r}$$

and this is the same thing as the magnitude of the acceleration, a:

$$a=\frac{\delta v}{\delta t}=\frac{v^2}{r} \quad \dots\dots\dots\dots\dots\dots(19)$$

Since δv is directed radially inwards, so is a. To produce such an inward acceleration requires a force F_i which from Newton's Second Law (equation 14) will be given by

$$F_i=\frac{mv^2}{r} \quad \dots\dots\dots\dots\dots\dots(20)$$

—the required expression for the centripetal force. This has the same form as the relation obtained by dimensional analysis.

Does equation 20 tie in with your experimental deductions? In practice does F_i really increase as m and v increase? Does F_i increase as r decreases?

So far the discussion has centred about the ground-based observer. He will be able to tell someone on a turntable just why he, the rider, has to keep pulling inwards to constrain the object to move in a circular path. Of course the person on the turntable may not appreciate that he is in an *accelerating* frame, although his stomach (or his ears) should normally tell him so. According to the person in the rotating frame of reference the mass keeps trying to accelerate outwards as if acted upon by some invisible force, the so-called *centrifugal* force F_0. To prevent the mass from moving outwards the person on the turntable applies an inwardly directed force which he knows will equal F_0 when the radial movement ceases (no net force means no apparent acceleration).**

centrifugal force

* *We represent the* magnitude *of* $\mathbf{\delta v}$ *by* δv *and the magnitude of* $\mathbf{v}$ *by* v.

***The person on the turntable has, we assume, 'learnt' his mechanics while in an inertial frame. In particular, he agrees to define zero force as that which produces no apparent acceleration and furthermore that the 'head-to-tail' rule defines how forces are to be added.*

In other words he deduces that the centrifugal force F_o is

$$F_o = F_i \qquad \text{or, from equation 20,}$$

$$F_o = \frac{mv^2}{r} \quad \dots\dots\dots\dots\dots\dots (21)$$

Although it adds nothing new to the physics of the situation, there are several alternative ways of writing equations 20 and 21. If T is the time required for m to complete one circle of rotation (the *periodic time*, or, simply, the *period*), then $v = 2\pi r/T$, enabling equation 19 to be written

period of rotation

$$a = \left(\frac{2\pi r}{T}\right)^2 \Big/ r$$

$$a = r\left(\frac{2\pi}{T}\right)^2 \quad \dots\dots\dots\dots\dots\dots (22)$$

It is customary to write $2\pi/T$ as ω (pronounced 'omega') and to call it the *angular velocity* or *angular frequency* of the particle. Making this substitution into equation 22

angular velocity
angular frequency

$$a = r\omega^2 \quad \dots\dots\dots\dots\dots\dots (23)$$

and, as an alternative form of equation 20

$$F_i = mr\omega^2 \quad \dots\dots\dots\dots\dots\dots (24)$$

After deriving any relation it is always worth while pausing to see if it 'makes sense'.

SAQ 13
Calculate the force necessary to keep a 500 g bag of sugar (about 1 lb.) revolving in a circle of radius 0.75 m with a period of 0.4 s.

The detailed working of the problem is given on page 60.

Bearing in mind that the weight of an average apple is 1 N, does your answer for the force seem reasonable? Had your answer come to, say, 0.092 N, as it could have by dropping a factor of 10^3 in the calculation, would you have been worried? If not, you should try the experiment.

You should now try *SAQ*s 14–16 (p. 55).

3.6 Momentum

3.6.1 Hidden forces

It is a good idea to remind yourself at this point of the primary quality of a force, which is, at least in our experiments, to accelerate things. Indeed our first experiments were devoted to studying how the velocity of an object changed with time under the influence of a force. We could watch the velocity change. We could see the spring pulling. Now, as you will learn in later Units, there are situations where we know that the velocity of an object has changed, yet we know none of the details about how it has been changed. In such situations we may know neither the magnitude of the force nor for how long it acted. Such a situation has been created in the experiment shown in Figure 11. Here, a puck moving at constant speed enters a tunnel at a known time to emerge at a later time with a different, but again constant, speed. What, we may ask, went on in the tunnel? The change in the puck's speed indicates that a force must have acted on it while it was hidden from view. But what force, and for how long did it act?

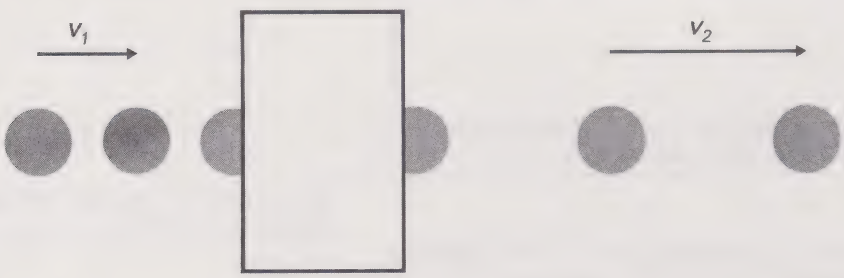


Figure 11 Showing a puck moving at constant speed entering a 'tunnel' at a known time, to emerge at a later time with a different, but again constant, speed. The diagram shows the puck at fixed time intervals.

Before we can attempt to answer these questions, we had better put down all the information we have. It is not much! We know that at one time, t_1, a puck of mass m, moving with a velocity v_1, disappeared from view only to reappear at a later time, t_2, moving at a new velocity v_2. We have

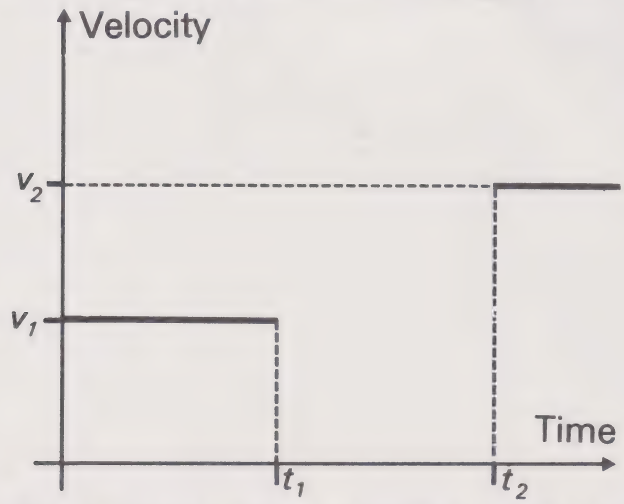


Figure 12 Showing graphically, the information contained in Figure 11, namely that at a known time, t_1, *a puck moving with a speed* v_1 *disappeared from view to emerge at a later time* t_2 *moving with a speed* v_2.

recorded this information in graphical form in Figure 12. Note, and this is important, the horizontal axis, the *abscissa*, represents time *not* distance; it tells us when, not where, the puck entered the tunnel.

Since we are literally in the dark as to what actually happened in the tunnel, we can only guess. Perhaps the simplest hypothesis is that during the interval t_1 to t_2 the velocity increased at a constant rate, i.e. with a constant acceleration. This guess is shown schematically in Figure 13a. If you have forgotten why a linear velocity-time graph signifies a constant acceleration, you should read Appendix 1 again. To produce a constant acceleration requires . . .

Requires what?

A constant force.

What force was acting on the puck before time t_1 and after time t_2?

None; it was moving at constant velocity. Figure 13b records the supposed forces acting on the puck. We can actually deduce the magnitude of the force, F say, which acted during the interval $t_2 - t_1$.

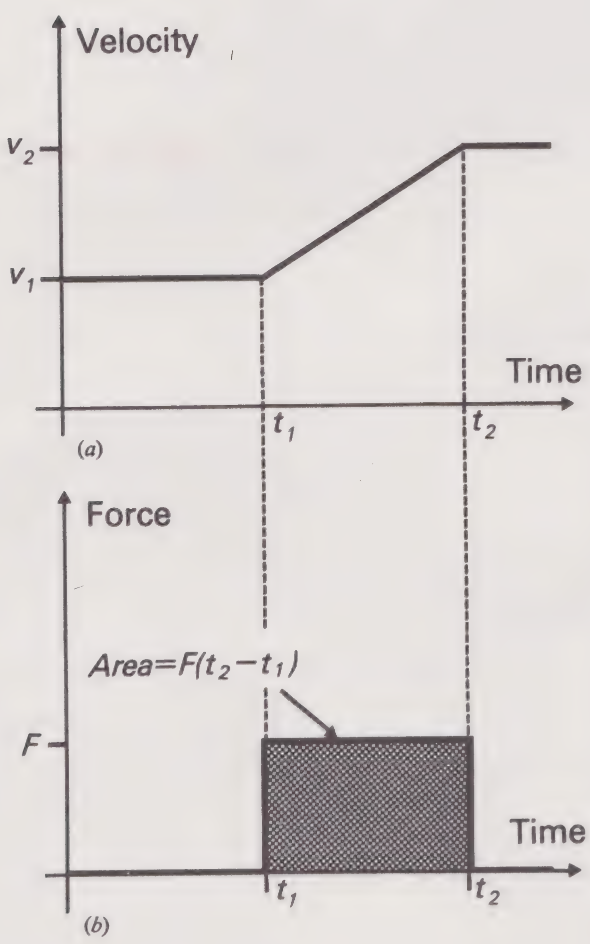


Figure 13 Shows one possible way that a puck might change speed while hidden from view. (a) Shows, graphically, the hypothesis that during the interval t_1 to t_2 the velocity increased at a constant rate. (b) The corresponding forces acting on the puck.

From equation 14

$$F = ma$$

$$= m\,\frac{(v_2 - v_1)}{(t_2 - t_1)}$$

Cross multiplying gives:

$$F\,(t_2 - t_1) = mv_2 - mv_1 \quad\ldots\ldots\ldots\ldots\ldots(25)$$

or

$$F\,(t_2 - t_1) = p_2 - p_1 \quad\ldots\ldots\ldots\ldots\ldots(26)$$

where we have written p for mv; this product mv is called the *momentum* of the object. Since velocity is a vector, the product mv is a vector quantity. If we wish to emphasize that we are dealing with vectors, equations 25 and 26 should be written as

momentum

$$\mathbf{F}\,(t_2 - t_1) = m\mathbf{v}_2 - m\mathbf{v}_1$$

or

$$\mathbf{F}\,(t_2 - t_1) = \mathbf{p}_2 - \mathbf{p}_1$$

Since the present discussion is limited to the situation where the object emerged from the tunnel moving in the same direction as when it entered the tunnel, the notation of equations 25 and 26 will be adequate.

What does the left-hand side of equation 25 or equation 26 represent in the force-time graph, Figure 13b?

It represents the area under the force-time graph. The height of the rectangle is F while the base is $(t_2 - t_1)$. So although we know that the momentum of the object changes from mv_1 to mv_2 we can deduce separately neither the force nor the time for which it acted. All we know is the product of the unknown force and the unknown time, or, as we say, the *impulse* of the force. To emphasize this uncertainty, look at Figure 14a, where we have

impulse of a force

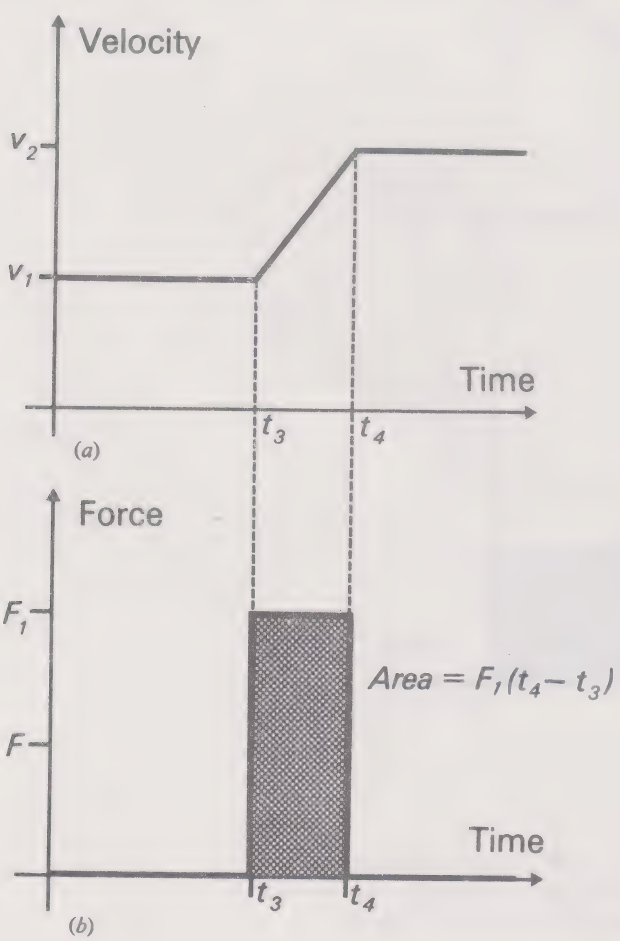


Figure 14 Another way in which the puck might change speed while hidden from view. (a) The acceleration is constant and occurs in the interval t_3 to t_4. (b) The corresponding forces acting on the puck.

assumed that the object is again accelerated at a constant rate but the acceleration is more rapid and for a shorter time. The constant force F_1 required is shown in Figure 14b.

Follow through exactly the same argument as before to deduce the value of the impulse of the new constant force, F_1, acting for time (t_4-t_3).

Once more $F_1\,(t_4-t_3)=mv_2-mv_1$. Again all we know is the value of the impulse; we can separately infer neither F_1 nor (t_4-t_3).

A 0.5 kg puck enters a tunnel moving at a speed of 2.0 m s^{-1} and emerges with a speed of 8 m s^{-1}. What constant forces might have acted on the puck and for how long? Give a few examples.

The change in momentum of the puck is

$$mv_2-mv_1=0.5\times 8.0-0.5\times 2.0$$
$$=3 \text{ kg m s}^{-1}$$

which equals the impulse of the force. So the force could have been for example 3 N for 1 s, 300 N for 10^{-2} s, or even 6×10^6 N for 5×10^{-7} s. There is no way of telling. Incidentally, you should convince yourself that the dimensions of impulse (N s) are the same as those of momentum (kg m s^{-1}). If they are not the same there is a mistake somewhere in the argument. (See section 4 in *HED*.)

Although we have assumed that the hidden force is constant, it is a general truth that, irrespective of how the force varies, the area under the force-time curve always equals the change in the object's momentum. If you are familiar with the ideas of calculus you may be able to prove this for yourself. The argument is to be found in Appendix 3 (Black).

The following example may make clear the practical importance of momentum changes and the associated impulses.

SAQ **17**
A 90 kg person is a passenger in a car travelling at 60 km per hour, which has a head-on collision and stops 'dead'. If the car seat belt holding the passenger 'gives' for 0.2 s calculate the force which the belt exerts on the passenger as he is brought to rest.

Assume the belt provides a constant force.

The detailed working of the problem is given on p. 61.

The answer is 7 500 N. Such a force, when applied, as it is, over a considerable area of the body is unlikely to cause significant damage. With no safety belt the dashboard may have to provide the impulse to reduce the passenger's momentum. Since a dashboard, or indeed an over-rigid seat belt, will have less 'give', the force exerted on the passenger will be larger. If the time for which the force acts is reduced say a thousand fold, its magnitude will be increased a thousand fold; the products of force and time being constant.*

* *Although this example of the car safety belt is frequently employed as a means of giving one a 'feel' for momentum, it actually fails to distinguish momentum from kinetic energy. As you will learn in Unit 4, if a constant force* F *acts through a distance* s, *on a body of mass* m, *moving at a speed* v, *so as to bring the body to rest, then* $Fs=\frac{1}{2}mv^2$. *(The product* $\frac{1}{2}mv^2$ *is called the kinetic energy of the body.) A seat belt which 'gives' will stretch through a* large distance s *and so provide a* small *force* F, *such that the product* Fs *is equal to* $\frac{1}{2}mv^2$. *The damage sustained by a person who is involved in an accident depends both on his momentum and on his kinetic energy. In fact there is no way of describing what momentum is in 'everyday speech'. Most of the descriptions more closely match* $\frac{1}{2}mv^2$ *than* mv. *Before long you will meet other concepts which like momentum can only be adequately described in mathematical symbols.*

3.6.2 Action-reaction forces and momentum conservation

If you have ever banged your fist on a table you will have noticed that whenever you bang the table it bangs you. You *act* on the table with a certain force and it *reacts* back with an apparently equal force on you. Place a book on a flat table and leave it alone; it leaves you alone—*action* equals *reaction*. Push on the book, i.e. act on the book, and it will push back on your fingers. Indeed it is the book's reaction which you sense. Push harder, so accelerating the book still more rapidly, and the reaction will be greater. Our subjective impression is that the reaction apparently equals the action. You might think we could prove the hunch by inserting two spring balances, one after another, between ourselves and the book. It is true that the balances give the same readings for the force but, how can a balance tell which end is being pushed, or is it pulled? We do not know whether a balance is measuring 'action' or 'reaction'. So although we cannot prove our hunch directly it does seem to be reasonable to assume that action equals reaction. This assumption is known as *Newton's Third Law*. As we shall see the prediction that momentum is reversed in collisions, a prediction made assuming that action equals reaction, is in accord with experiment. So Newton's Third Law may be verified indirectly.

action and reaction

Newton's Third Law

To see the consequences of action-reaction forces, let us consider a head-on collision between two magnetic pucks. These are pucks made of ring

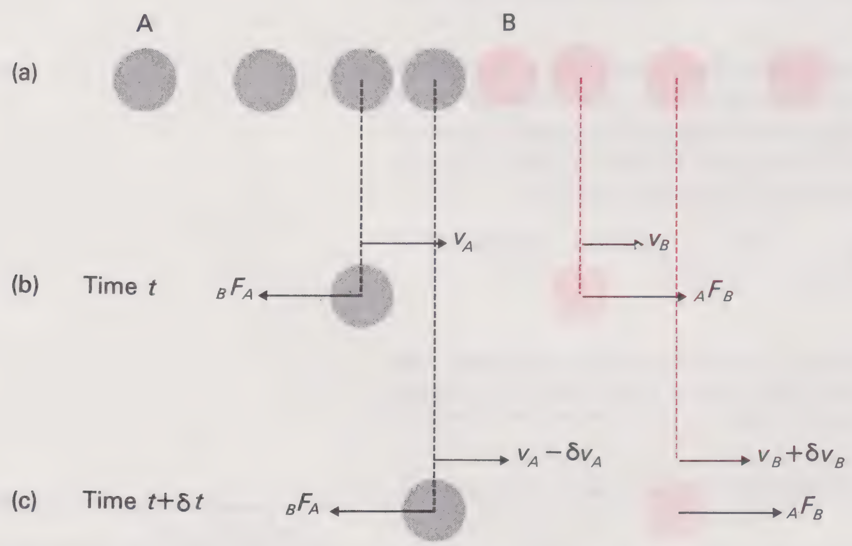


Figure 15 Action-reaction forces. (a) A puck A is shown closing in on an initially more slowly moving puck B. (b) The forces acting on each puck and the corresponding velocities at time t. *(c) The situation at a later time* $t+\delta t$.

magnets which repel each other over appreciable distances. Our argument in no way depends on the pucks being magnetic; we employ them solely to produce a gentler collision. (You will see a collision of this kind in this Unit's TV programme.) Figure 15a shows puck A, of mass m_A, closing in on an initially more slowly moving puck B, of mass m_B. If we think about the situation at a time t, as shown in Figure 15b, A will be acting on B with a force which we may write as $_AF_B$ (meaning A's push on B) while B will be reacting back on A with force $_BF_A$ (meaning, of course, B's push on A). The effect of $_AF_B$ will be to speed up B, while $_BF_A$ will slow down A. Anyone who has ever bumped into another person will know this much! To find out more about the changes consider the situation at a slightly later time, say $t+\delta t$, shown in Figure 15c. During the short time interval δt both $_AF_B$ and $_BF_A$ will be effectively constant; their effect will have been to increase the velocity of B by δv_B, say, and to decrease that of A by δv_A.

Can you write down expressions relating the momentum changes in A and B to the impulses?

Applying equation 25 we find that

$$_{B}F_{A}\,\delta t = m_{A}\delta v_{A} \quad \ldots\ldots\ldots\ldots\ldots(27)$$

$$_{A}F_{B}\,\delta t = m_{B}\delta v_{B} \quad \ldots\ldots\ldots\ldots\ldots(28)$$

In view of Newton's Third Law, the left-hand sides of equations 27 and 28 are equal.

Therefore, $$m_{A}\delta v_{A} = m_{B}\delta v_{B}$$

As mass and the product of change in velocity is just the change in momentum we can write

$$\delta p_{A} = \delta p_{B}$$

Remembering that B has speeded up while A has slowed down, we have proved that the momentum gained by B equals that lost by A, or taken together the total momentum of A and B has not changed during the time between t and $t+\delta t$. As there is nothing special about the time interval we have examined, the same argument may be applied throughout the collision sequence. The total momentum before the collision should equal the total momentum after the collision, i.e. momentum should be *conserved*. But is it?

conservation of momentum

Find two objects of the same mass, such as a couple of similar paperback books. Place one of these on a smooth flat surface, e.g. a laminate-covered kitchen table. Give the other a shove, so that it collides head-on with the stationary one. Notice how they swop places; the moving one stops dead after the collision, the one that was stationary before moves off with the velocity which the incoming one had at the moment of impact. If you are familiar with the novelty called 'Newton's Cradle', the result will not be new to you. A billiard player also knows that when a billiard ball collides head-on with a stationary one the two balls exchange roles. Remembering that momentum is properly a vector quantity, give a vector representation to the experiment, showing that the total vector momentum before and after the collision are the same.

Your vector representation should look like that of Figure 16 although the directions may be different depending on your line of shot. If your collision is not head-on the tracks will make finite angles with each other. But even in the case of oblique collisions momentum is conserved—provided momentum is given its proper vectorial representation, i.e. vectors are drawn of magnitude mv and in the direction of v.

Figure 16 is on p. 44.

What happens to the momentum of a sliding book when it eventually slows down and stops?

Suppose the books had been projected in a West to East direction, i.e. in the same sense as the Earth's rotation.

Horizontal forces of friction between the table and the book are acting to slow the book down. At the same time, the book is reacting back on the table with horizontal forces which go to speed up the table and hence speed up the Earth's rate of rotation. Momentum is transferred from the book to the Earth.

Where did the book's momentum come from in the first place?

As you pushed on the book, it pushed back on you through your arms, to your feet, to the Earth. As you pushed the book forward it pushed the

Earth backwards. The momentum gained by the book presumably equals that lost by the Earth (as its rate of forward rotation was reduced). By the time the book had stopped moving it had given its momentum back to the Earth.

Assuming both Newton's Second and Third Laws, the conservation of momentum has been predicted. Experimentally, momentum has been found to be conserved. Since Newton's Second Law has been established independently it is tempting to conclude that Newton's Third Law is *therefore* universally correct. There could of course be other ways of accounting for momentum conservation.

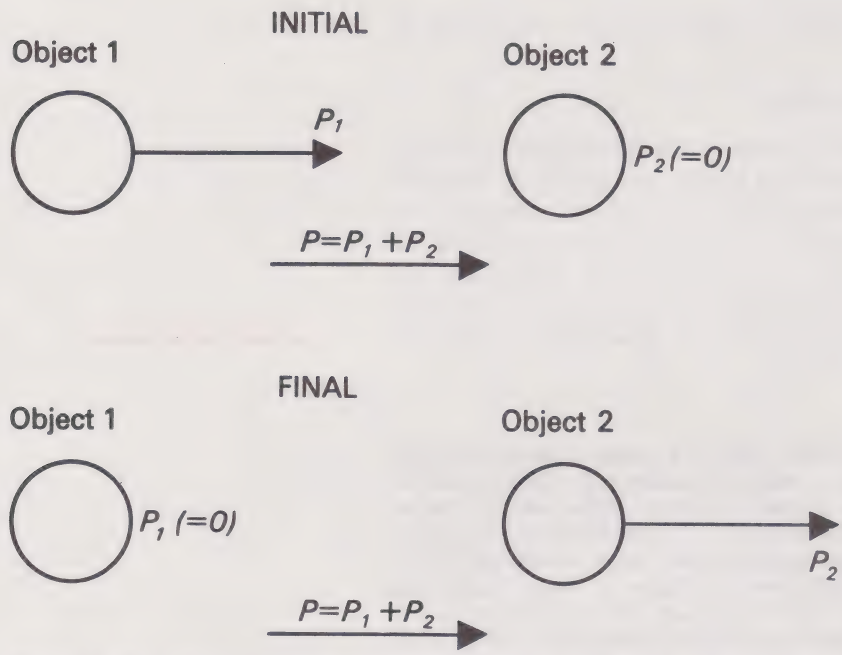


Figure 16 Illustrating momentum conservation in a head-on collision between two objects of the same mass, one of which was initially at rest. p_1 *represents the momentum of the incoming object and* p_2 (=0) *the momentum of the object at rest. Underneath is shown the total initial momentum,* p. *As a result of the collision the incoming object has come to rest* (p_1=0) *while the object initially at rest moves off with momentum* p_2 *equal to that of the incoming object. The total final momentum is the same as the total initial momentum.*

3.6.3 Relativistic momentum

Throughout this discussion of momentum we have failed to state whether we have been talking about proper or improper velocity, or, if you prefer, whether the clocks used in measuring the velocities of the objects travel with the objects, or are stationary beside the scale measuring the distance travelled. As we saw in section 3.3, such carelessness makes little difference unless the speeds approach that of light. But once we attempt to discuss the behaviour of such fast moving objects, we must state clearly which velocity is meant. Normally, of course, improper velocities are measured; in any practical set up the clocks will be alongside the scale. If we check up on whether momentum, as defined by mass × improper velocity is conserved in collisions between, say, sub-nuclear particles moving at speeds approaching that of light, we find that the law of conservation of momentum breaks down. However, the conservation law does still hold *provided* momentum is defined as

$$p = m_0 v_{\mathrm{im}}/\sqrt{1-(v_{\mathrm{im}}^2/c^2)} \ldots\ldots\ldots\ldots\ldots (29)$$

where m_0, known as the *rest mass* of the object, is the mass that is measured at very low speeds, i.e. what, up to now, we have been calling m. At such low speeds, where v_{im} is very much less than c, the ratio v_{im}/c is negligible **rest mass**

compared to unity and equation 29 reduces to $m_0 v_{im}$, which is our original definition of momentum. (Remember, in our previous formulation of mv, m was the mass at low speeds which should be written m_0, and v was actually v_{im}.)

There are two ways of interpreting equation 29. One way is to combine the denominator with the v_{im} factor in the numerator, thus:

$$p = m_0 \times \frac{v_{im}}{\sqrt{1 - v^2_{im}/c^2}}$$

Now, from equation 6

$$v_{pr} = \frac{v_{im}}{\sqrt{1 - v^2_{im}/c^2}} \quad \text{...........(6)}$$

In this way, momentum is defined simply as the product of the constant mass m_0 and the proper velocity:

$$p = m_0 v_{pr} \quad \text{.................(30)}$$

An alternative interpretation of equation 29 arises if we combine the denominator with m_0 factor, and call this the *relativistic mass*, m, of the moving object, thus:

relativistic mass

$$m = \frac{m_0}{\sqrt{1 - v^2_{im}/c^2}} \quad \text{...........(31)}$$

In this approach momentum is defined as the product of this variable mass, m, and the improper velocity:

$$p = m v_{im} \quad \text{............ ...(32)}$$

It doesn't matter in the least which of these two definitions we choose for p. Experiments simply tell us the value of p; we may interpret this experimental value as we wish. The second viewpoint, summarized in equations 31 and 32, is perhaps the more fashionable, and it is the one we shall adopt in later Units. If one makes a series of measurements of the momentum of an object travelling at various speeds (v_{im}), then, according to this second interpretation, equation 31, rewritten as $m = p/v_{im}$, can be used to calculate the relativistic mass, m, at these different speeds. Figure 17 shows the results of such a study on the electron. (The momentum, p, is found by measuring the extent to which the electron's path is bent in a magnetic field.) Here the deduced values of m are expressed as fractions

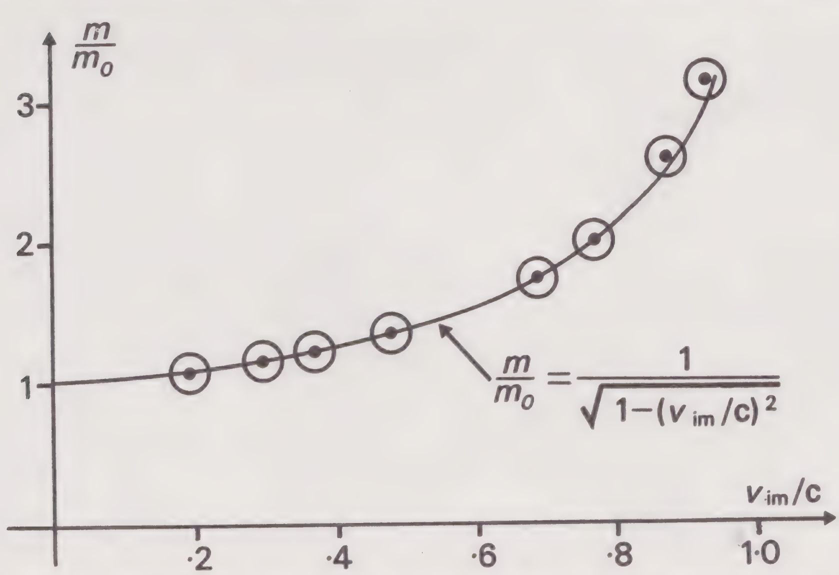


Figure 17 Showing how the relativistic mass m *of an electron varies with speed* v_{im}. *Here the values of* m *are expressed as fractions of the rest-mass*, m_0, *while the speeds* v_{im} *are expressed as fractions of the speed of light*, c. *Superimposed on the experimental points is the variation to be expected from equation* 31.

of the rest mass, m_0, while the speeds, v_{im} are expressed as fractions of the speed of light, c. Superimposed on the experimental points is the variation of m/m_0 to be expected from equation 31. Of course, had we adopted the first interpretation of p, equation 30 rewritten as $v_{pr}=p/m_0$ would have told us how the proper velocity of the electron varied with the improper velocity. The experiments also demonstrate that p increases with v_{im}.

***SAQ* 18**

A futuristic rocket ship with a rest mass of 10^6 kg is moving with an improper velocity of 0.9 c. (Take the velocity of light c as 3.0×10^8 m s^{-1}.) What is the momentum of the ship and its relativistic mass?

The detailed working of the problem is given on p. 61.

The momentum is 6.2×10^{14} kg m s^{-1}. The relativistic mass is 2.3×10^6 kg, which is 2.3 times greater than the rest-mass.

You should now do *SAQ*s 19–20, p. 56.

3.7 Recapitulation

We started off this Unit by making a study of some of the fundamental properties of electromagnetic waves; a study provoked by the desire to find the fastest possible method of transmitting information. We learnt that the velocity of light depends on the medium through which it travels, the velocity being greatest in a vacuum, and that the measured velocity of light is independent of the velocity of the light source and of the velocity of the observer. A logical consequence of these assumptions is that a clock mounted on a moving object will measure a shorter time, called the proper time, to cover a certain distance than will two clocks set out alongside the distance scale. The clocks alongside the scale measure the so-called improper time. However, we showed that the difference between the two timings can be ignored unless the speed of the object approaches that of light.

A systematic study of how the acceleration of an object depended on the accelerating force and on the mass of the object led to Newton's Second Law, formulated first in arbitrary units, then in SI units. It was possible to account for the behaviour of objects located in a rotating framework and to derive expressions for the centrifugal and centripetal forces. Newton's Second Law was also employed in the discussions of weight.

The problem of how to discuss the behaviour of an object which changes speed under the influence of hidden forces led to the formulation of momentum. Assuming Newton's Third Law it was proved that momentum should be conserved in collisions—a prediction borne out by experiment. However, for momentum to be conserved in collisions involving objects moving at speeds approaching that of light it must be given a more general formulation than mv. This general formulation was shown to predict that the mass of an object should increase as its speeds approach that of light, as is observed in practice.

Appendix 1

Appendix 1 (Red)

Describing how things move

To most people, mention of the word 'velocity' or 'speed' would probably lead them to picture a car's speedometer. If asked what 54 m.p.h. meant they would probably, and correctly, reply that if the driver keeps going along so that the needle stays on the 54 m.p.h. mark he will cover 27 miles in ½ an hour, 54 miles in 1 hour, 81 miles in 1½ hours, etc. If asked how such a speedometer could be calibrated they might reply, although many are reluctant to invert the argument, that the car could be driven for, say, half an hour keeping the speedometer needle on some as yet unnamed point, and the total distance travelled could be measured. Had the car gone, say, 49 miles in 0.5 hour, then its speed was $49/0.5=98$ m.p.h. The reluctance to suggest such a technique may stem from knowing the problem of finding a free enough stretch of road where the car could be driven uniformly for anything like half an hour! But why not keep going for only ¼ hour (requiring a free stretch of 24½ miles), or for 2 minutes or even 1, and measure the distances? The only possible objection is that it is difficult to measure these small distances, but cameras set up alongside the road to photograph the car's position at intervals of say 1 s, 0.1 s, or even 0.01 s will dispose of this objection (a scale set out along the road could be photographed simultaneously).

In fact the only way to be really certain of a car's speed is to measure the distance δs, covered in a short time δt (the symbol δ pronounced 'delta' just means 'a small amount of'). By division the speed of the car is

$$v=\frac{\delta s}{\delta t} \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

The shorter the chosen value of δt, and hence the smaller the distance δs gone, the more precise the idea of velocity becomes.

Using equation 1, the car's speedometer can now be fully calibrated.

Suppose we go on a journey reading the car's speed at various times as we go along. A good way to present the information is to plot a graph of speed against time. Such a plot might look like that shown in Figure 18. On this particular journey the car started off in town, then travelled along a motorway, but had an accident. After it was repaired, the driver proceeded

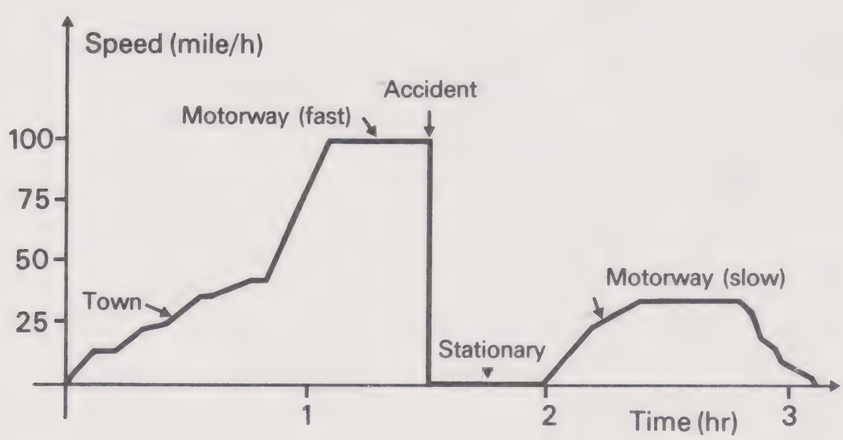


Figure 18 Showing how the speed of a motor car might vary with time.

more cautiously along the motorway. Finally he encountered some congested traffic. You should satisfy yourself that these seem reasonable interpretations of Figure 18.

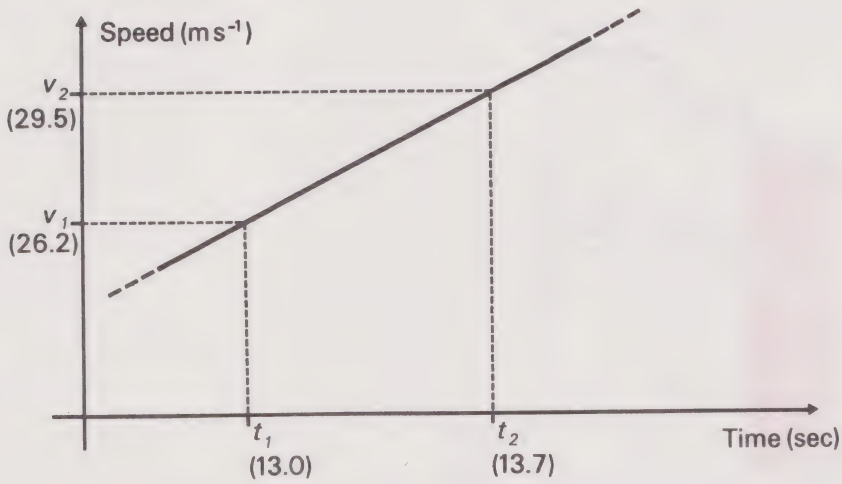


Figure 19 Illustrating the particular case where the velocity of a car increases linearly with time.

Rather than discuss this general case in more detail imagine that the plot was actually linear, as shown in Figure 19. Such a graph might possibly be obtained in practice if the driver kept his foot on the accelerator.

If you obtain a straight-line (i.e. a linear) graph, one of the first things you think about calculating is its slope. To determine the slope two lines may be drawn in, parallel to the velocity axis, starting at times t_1 and t_2. The corresponding velocities v_1 and v_2 may be read off from the graph. (Sample numerical values are included in Figure 19.) As the time changed from t_1 to t_2 (from 13.0 s to 13.7 s) the velocity changed from v_1 to v_2 (from 26.2 to 29.5 m s^{-1}). In other words the velocity changed by (v_2-v_1) (=3.3 m s^{-1}) in time (t_2-t_1) (=0.7 s). Therefore the velocity change per unit time, i.e. the acceleration, a, is given by

$$a=\frac{v_2-v_1}{t_2-t_1}$$

In the particular example given, a=3.3 m s^{-1}/0.7 s=4.7 m s^{-2}. You will see that a is the slope of the graph, which is constant throughout in a linear graph. Of course in the more general case shown in Figure 18 the slope, i.e. the acceleration, kept changing. For acceleration to have meaning in such a general case one must only examine the velocity change, δv occurring in a small time interval δt. Then

$$a=\frac{\delta v}{\delta t}$$

The smaller you make δt, the smaller become the possible wild fluctuations in δv, and the more meaningful becomes the acceleration a.

Another routine quantity to evaluate from graphs is the area underneath them. In the case of the graph showing constant acceleration we will evaluate the area under the straight line between time 0 and t. With this

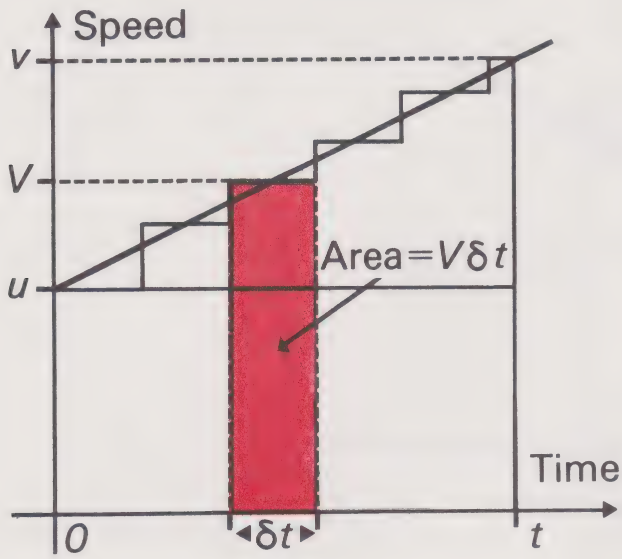


Figure 20 *Illustrating that the distance travelled by a car is the area under the graph of velocity plotted against time. We used the symbol* V *for the particular value of the velocity* v *at which we calculated the area* Vδt *under one step.*

end in view Figure 19 has been redrawn as in Figure 20; the original straight line has been replaced by a series of discrete steps. This stepped graph is constructed so that the area under it is the same as that under the original graph; the area under the latter is the sum of the areas under all the steps. (One is shown shaded.) Now the area under one of these steps is $v\delta t$, which is nothing more than *distance* gone in this time interval δt. Hence the total area between zero time and t represents the total distance gone in this period.

I.e. in time t distance gone = Area under velocity-time curve between time zero and t.

It is not hard to see that this area can be written in terms of the velocity at time zero (u, say) and that at time t (v, say). Thus

$$\begin{aligned} s = \text{Area} &= \text{Area of rectangle} + \text{area of triangle} \\ &= (ut) + \tfrac{1}{2}\,(\text{base} \times \text{height})^* \\ &= ut + \tfrac{1}{2}t\,(v-u) \end{aligned}$$

i.e.

$$s = \frac{(u+v)t}{2} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

* *See* MAFS, *section 2.A.1.*

This result can be put into different forms on recalling that the acceleration is the slope of the line (and constant).

I.e. $$a=\frac{(v-u)}{t}$$

i.e. $$at=v-u$$

or $$v=u+at \quad \text{.................}(5)$$

Substituting equation 5 into equation 4 gives

$$s=\frac{(u+u+at)t}{2}$$

i.e. $$s=ut+\tfrac{1}{2}at^2 \quad \text{................}(6)$$

An alternative expression, involving only v, u, and a, may be obtained by substituting the value of t given by equation 5, namely $t=(v-u)/a$, into equation 4.

$$s=\frac{(v+u)\,(v-u)}{2a}$$

$$s=\frac{v^2-u^2}{2a} \quad \text{..................}(7)$$

So now we have various alternative, but essentially identical, relations between the distances travelled in a time, t, by a body which was moving with a velocity u at time zero and accelerating with a constant acceleration a.

Where was the assumption made that the acceleration was constant?

The graph was supposed to be linear. The areas evaluated in equations 4, 6 and 7 are the areas under straight line graphs. Conversely if a body moves so as to obey equation 5, 6 or 7 its acceleration must be constant.

Appendix 2

Dimensional analysis applied to centripetal forces

Everyday experiences suggest that the inwardly directed force F_1 required to keep an object rotating in a circular path at, say, the end of a spring depends on the mass m of the object, the radius r of its path and the speed v at which it is moving. Other factors such as, for example, the shape of the object are of no obvious importance. If it should not prove possible to make the dimensions of F_1 agree with those of the product of m, v, and r, each raised to an appropriate power, then other factors will have to be considered. Even if agreement is obtained, it is still possible that other factors have been neglected—dimensional analysis may not yield a unique solution. But assuming

$$F_1 = \text{constant} \times m^{\alpha} v^{\beta} r^{\gamma}$$

Putting in dimensions of force, mass, velocity and radius, gives

$$MLT^{-2} = M^{\alpha}\left[\frac{L}{T}\right]^{\beta} L^{\gamma}$$

—force, remember, has the dimensions of mass × acceleration (Newton's Second Law).

Equating the powers of M, L, and T on the right-hand side with those on the left-hand side gives:

$$\text{for mass } \alpha = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{for length } 1 = \beta + \gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{for time } -2 = -\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\therefore \quad F = \text{constant} \times \frac{mv^2}{r}$$

We have now combined m, v, and r in the only way that the units of the combination are those of force. Dimensional analysis is, of course, powerless to indicate what the dimensionless constant might be.

Appendix 3 (Black)

Momentum—a more general discussion

Newton's Second Law, equation 15, relates the acceleration a produced in a body of mass m by a force F, viz.

$$F=ma$$

But acceleration, being the rate of change of velocity can be written as

$$a=\frac{dv}{dt}$$

$$\therefore \quad F=m\frac{dv}{dt} \text{ or, provided } m \text{ is a constant,}$$

$$F=\frac{d}{dt}(mv)$$

or

$$F=\frac{dp}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where the substitution $p=mv$ has been made, p being the momentum of the body.

Cross-multiplying equation (1) gives

$$Fdt=dp \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating equation (2) gives

$$\int Fdt=\int dp \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

If a force acts during the interval t_1 to t_2 as a result of which the momentum of the body changes from p_1 to p_2 then

$$\int_{t_1}^{t_2} Fdt = \int_{p_1}^{p_2} dp$$

$$\int_{t_1}^{t_2} Fdt = \Big[p\Big]_{p_1}^{p_2}$$

$$\therefore \int_{t_1}^{t_2} Fdt = p_2 - p_1 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The left-hand side of equation 4 is the area under a graph showing how the force acting on the body changes with time, between times t_1 and t_2. This so-called impulse of the force is, as equation 4 demonstrates, equal to the resulting change in the momentum of the body.

Self-Assessment Questions

Section 3.2

Question 1 (*Objective 2*)

Which, if any, of the following properties of light are assumed in formulating the theory of special relativity?

1 The velocity of light exceeds that of sound.
2 The wavelength of blue light is less than that of red light.
3 The velocity of light does not depend on the speed of its source.
4 No optical experiments performed within a laboratory have ever shown up any uniform motion of the laboratory.

Section 3.3

Question 2 (*Objective 3*)

The experiment described in section 3.3.1 is performed in a car with $L = 1.5$ m moving at a speed of 2.0×10^8 m s^{-1}. Calculate the proper and improper times for the light to traverse the path from the bulb to the mirror and back to the clocks.

Question 3 (*Objective 3*)

In 1965 Ron Clarke ran the 10 000 metre event in 27 mins. 39.4 secs. Had Clarke carried a watch, how would his timing have differed from the judges? Needless to say the judges were stationary alongside the track.

Question 4 (*Objective 3*)

The proper velocity of an object:

1 Is greater than the improper velocity of the object.
2 Is less than the improper velocity of the object.
3 Is equal to improper velocity of the object.
4 Is equal to the speed of light in a vacuum.

Section 3.4

Question 5 (*Objective 5*)

In an inertial frame:

1 Newton's First Law holds true but optical experiments enable one to detect that the frame in which the experiments are performed is moving.
2 Newton's First Law holds true.
3 Everything behaves in a sluggish fashion.
4 Newton's First Law holds true, provided that the speed is much less than the speed of light.

Section 3.5

Question 6 (*Objective 4*)

A car accelerates away from rest at 4 m s^{-2}. What is its velocity after it has gone 8 m?

Question 7 (*Objective 4*)

A sprinter who is travelling at 2 m s^{-1} accelerates at a constant rate while he travels 8 m. His final velocity is 5 m s^{-1}. How long does he spend accelerating?

Question 8 (*Objective 4*)

A puck accelerates from rest at a constant rate for 3 s through a distance of 5 m. What is the puck's acceleration?

Question 9 (*Objective 5*)

Make an order of magnitude estimate of the push that human legs can provide while accelerating away from rest.

Question 10 (*Objective 5*)

A car of mass 400 kg accelerates away at a constant rate from rest. In 15 s it has reached a speed of 50 km per hour. What force is the engine providing?

Question 11 (*Objective 5*)

An ascending escalator set at 45° to floor level is moving at 0.8 m s^{-1}. Someone is walking up the escalator at 1.1 m s^{-1}. Use the head-to-tail rule to deduce his speed and direction relative to someone stationary at the foot of the escalator.

Question 12 (*Objective 5*)

What roughly is the force of attraction between the Earth and a 'quarter pound' slab of chocolate? You should know that, when released, a slab falls with a constant acceleration. In an actual experiment a slab in falling from rest through a distance of 3 m acquired a final speed of 7.5 m s^{-1}.

Question 13 (*Objective 5*)

Calculate the force necessary to keep a 500 gm bag of sugar (about 1 lb.) revolving in a circle of radius 0.75 m with a period of 0.4 s.

Question 14 (*Objectives 5*)

In the SI system of units:

1 Experiment shows that the force required to accelerate a mass of 1 kg at a constant rate of 1 m s^{-2} is 1 N.
2 The weight of 1 kg is 1 N.
3 Densities are measured in kg m^{-3}.
4 Centripetal forces are measured in kg m s^{-1}.

Question 15 (*Objective 5*)

An inwardly directed force is required to keep an object moving round in a circular path because:

1 The velocity of the object keeps changing.
2 The speed of the object keeps changing.
3 The acceleration of the object keeps changing.
4 The mass of the object depends on its orientation in space.

Question 16 (*Objective 5*)

The angular frequency, ω, of an object traversing a circular path of radius r in a periodic time T is defined as:

1 $2\pi r/T$.
2 $2\pi/T$.
3 $2\pi/rT$.
4 $2\pi T/r$.

Section 3.6

Question 17 (*Objective 6*)

A 90 kg person is a passenger in a car travelling at 60 km $hour^{-1}$, which has a head-on collision and stops 'dead'. If the car seat-belt holding the passenger 'gives' for 0.2 s calculate the force which the seat-belt exerts on the passenger as he is brought to rest. Assume the belt provides a constant force.

Question 18 (*Objective 6*)

A futuristic rocket ship with a rest mass of 10^6 kg is moving with an improper velocity of 0.9 c. (Take the velocity of light c as 3.0×10^8 m s^{-1}.) What is the momentum of the ship and its relativistic mass?

Question 19 (*Objective 6*)

In a collision between two objects:

1 the total momentum before and after the collision is always the same;
2 the total momentum before and after the collision is never the same;
3 the total momentum before and after the collision is sometimes the same;
4 momentum is not conserved but the rate of change of momentum is conserved.

Question 20 (*Objective 6*)

As the speed of an object increases towards that of light, its relativistic mass:

1 Falls towards zero.
2 Increases towards infinity.
3 Remains constant.
4 Oscillates sinusoidally.

Self-Assessment Answers and Comments

Question 1

Answers 3 and 4 are correct. See section 3.2.3.

Question 2

The proper time t_{pr} is the time the light flash takes to travel to the mirror and back (a distance of $1.5+1.5=3.0$ m) as judged by a passenger in the car. Since to him light travels at 3.0×10^8 m s^{-1}, the time taken is

$$t_{pr}=\frac{\text{distance}}{\text{velocity}}$$

$$=\frac{3.0\text{ m}}{3.0\times10^8\text{ m s}^{-1}}$$

$$\therefore\quad t_{pr}=1.0\times10^{-8}\text{ s.}$$

During the time $\frac{1}{2}t_{pr}$ the car travels a distance as measured by an observer on the road of (the car's speed) $\times\ \frac{1}{2}t_{pr} = 2.0\times10^8\times0.5\times10^{-8}\text{ s} = 1\text{m}$. This corresponds to the halfway stage shown in Figure 2. This calculation assumes that the car's speed that is specified is 'the distance gone as measured alongside the road divided by the time for the journey as measured on the car'. The distance travelled by the flash as measured by roadside observer during the interval is therefore $\sqrt{1^2+1.5^2}$ (in terms of Fig. 2 it is, applying Pythagoras' theorem $=\sqrt{L^2+(l/2)^2}$) which is $\sqrt{3.25}=1.8$ m. Therefore the total path length of the flash as seen by the ground-based observer is $2\times1.8=3.6$ m. Since light also travels at 3.0×10^8 m s^{-1}, to this observer the improper time, t_{im}, is

$$t_{im}=\frac{3.6\text{ m}}{3.0\times10^8\text{ m s}^{-1}}$$

$$t_{im}=1.2\times10^{-8}\text{ s.}$$

Question 3

Equation 6 shows that:

$$t_{pr}=t_{im}\sqrt{1-v^2_{im}/c^2}$$

Question 1 pinpointed the reason why t_{pr} is shorter than t_{im}.

Here t_{im}, the timing of the stationary judges who are separated by a distance of 10^4 m, is 27 mins. 39.4 secs. $=(27\times60)+39.4\text{ s}=1\,659.4$ s,

i.e. $\qquad t_{im}=1\,659.4$ s.

In the judges' opinion Clarke ran at a speed of v_{im}, given by

$$v_{im}=\frac{\text{distance measured by stationary judges}}{\text{judges' timing}}$$

$$=\frac{10^4\text{ m}}{1\,659.4\text{ s}}$$

$$\therefore\quad v_{im}=6.035\text{ m s}^{-1}.$$

Substituting $v_{im}=6.035$ m s^{-1} and $c=3.0\times10^8$ m s^{-1} into equation 6 gives

$$t_{pr}=t_{im}\sqrt{1-(6.0)^2/(3\times10^8)^2}$$

(It is quite adequate to take $v_{im}=6.0$ m s^{-1}, because the second term under the root sign is so much smaller than unity.)

i.e. $\qquad t_{pr}=t_{im}[(1-(6.0)^2/(3\times10^8)^2)^{\frac{1}{2}}]$

According to the binomial theorem (equation 3) when x is very much less than unity

$$(1+x)^m \approx 1+mx+\text{second-order terms.}$$

Here $x=-(6.0)^2/(3\times 10^8)^2$ and $m=\frac{1}{2}$.

So, to a very good approximation

$$t_{pr} \approx t_{lm}\,[1-\tfrac{1}{2}((6.0)^2/(3\times 10^8)^2)]$$

$$\approx t_{lm}\left(1-\frac{36}{18\times 10^{16}}\right)$$

$$\approx t_{lm}\left(1-\frac{2}{10^{16}}\right)$$

In other words t_{pr}, Clarke's own estimate of the timing would be less than that of the judges by 2 parts in 10^{16}.

Question 4

Answer is 1. Note that t_{pr} is always less than t_{lm} unless $v_{lm}=0$. Refer back to equation 6 in section 3.3.2.

Question 5

Answer is 2. See section 3.4.

Question 6

The information given is the initial speed u $(=0)$, the acceleration $a\,(=4\text{ m s}^{-2})$ and the distance gone $s\,(\doteq 8\text{ m})$. What is required is the final velocity v.

Equation 7 of the Appendix 1 shows that

$$v^2-u^2=2as$$

or $\quad v^2=2as+u^2$, or substituting for u, a, and s

$$v^2=(2\times 4\times 8)+0=64\text{ m}^2\text{ s}^{-2}$$

$$\therefore\quad v=8\text{ m s}^{-1}.$$

If you do not know how equation 7 was arrived at, read pages 49–51 again.

Question 7

Here $u=2\text{ m s}^{-1}$, $v=5\text{ m s}^{-1}$, and $s=8$ m. The problem is to find the time spent accelerating at a constant rate.

Equation 4 of Appendix 1 is relevant:

$$s=\left(\frac{u+v}{2}\right)t$$

$$\therefore\quad t=\frac{2s}{u+v}=\frac{16\text{ m}}{(2+5)\text{ m s}^{-1}}$$

$$=\frac{16}{7}\text{s}$$

$$\therefore\quad t=2.3\text{ s}.$$

A useful test of your understanding of equation 4 is to ask yourself where the factor of 2 originates.

Question 8

Here $u=0$, $t=3$ s and $s=5$ m. The problem is to find the acceleration a.
Equation 6 of the Appendix 1 is the relevant relation.

$$\text{i.e. } s=ut+\tfrac{1}{2}at^2$$

$$\text{or } a=\frac{2(s-ut)}{t^2}$$

Substituting for u, s and t gives:

$$a=\frac{2(5-0)}{3^2}\frac{m}{s^2}$$

$$=\frac{10}{9}\text{ m s}^{-2}$$

$$\therefore \quad a=1.1\text{ m s}^{-2}$$

Question 9

An average sort of mass to assume for an adult is around 10 stone, i.e. 140 lbs, i.e. $140/2.2 \approx 60$ kg; let us say 10^2kg. One might be able to sprint up to a speed of about the same as that kept up by a four minute miler! i.e. a top speed of around $1500\text{m}/4\times 60\text{s} \approx 6\text{m s}^{-1}$ (1 mile $\approx$ 1500m). Conceivably it would take 2 s to 6 s to reach this speed, i.e. one's acceleration is of order $6\text{ m s}^{-1}/4\text{ s} \approx 1\text{ m s}^{-2}$. Applying equation 14 gives as the force F required to produce this acceleration, a, in a body of mass m of 10^2 kg as

$$F=ma$$

$$=10^2\times 1\text{ kg m s}^{-2}$$

$$F=10^2\text{ N}$$

Since a leg muscle might well be able to lift up to 10^2 bars of chocolate, each of weight 1 N, it seems a plausible enough answer.

Question 10

The car's constant acceleration must first be determined from the information that it started from rest, $(u=0)$ and after a time t of 15 s it had reached speed v of 50 km per hour. The defining relation for acceleration is $a=(v-u)/t$ but you get an answer in mixed units involving hours and seconds if you simply substitute $u=0$, $v=50$ km per hour, and $t=15$ s. We should convert 50 km per hour into units of m s^{-1}.

$$\frac{50\text{ km}}{1\text{ hour}}=\frac{50\times 10^3\text{m}}{1\times 60\times 60\text{s}}$$

$$=\frac{50\times 10^3\text{ m}}{3.6\times 10^3\text{ s}}$$

$$=\frac{50}{3.6}\text{ m s}^{-1}$$

$$\therefore \quad a=\frac{50}{3.6\times 15}\,\frac{\text{ms}^{-1}}{\text{s}}=0.925\text{ m s}^{-2}$$

Applying Newton's Second Law, equation 14 gives the accelerating force provided by the engine as $400\text{ kg}\times 0.925\text{ m s}^{-2}=370\text{ kg m s}^{-2}=370\text{ N}$.

Question 11

The individual vectors to be added are shown in Figure 21, where lines have been drawn of length proportional to 0.8 m s^{-1} and 1.1 m s^{-1} respectively in the required directions. Putting the head of one vector at the tail of the other and then joining the first head to the last tail as in the figure gives the resultant. To a ground-based observer, the person on the escalator is moving up at 1.9 m s^{-1} at 45° to the ground. If you are still uncertain about how vectors should be added, read *MAFS*, section 4.D.

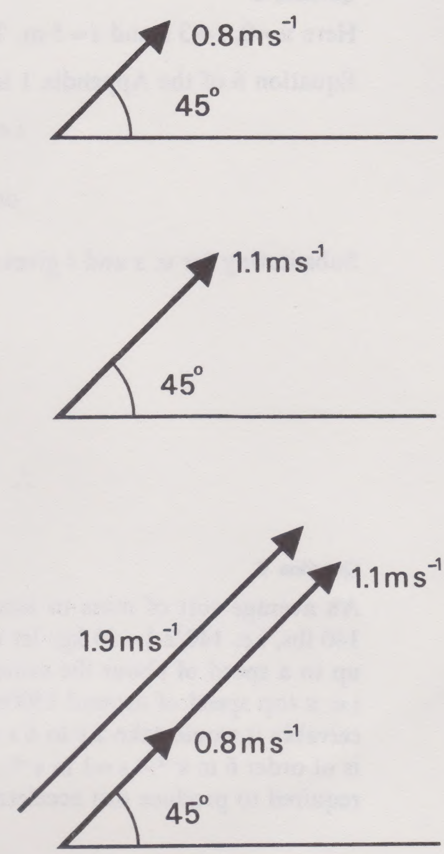


Figure 21 Showing how two vectors representing velocities of 0.8 m s^{-1} *and* 1.1 m s^{-1} *are combined by the 'head-to-tail' rule to give a resultant of* 1.9 m s^{-1}.

Question 12

First the acceleration of the bar of chocolate to the ground must be calculated. As we know the mass ($\frac{1}{4}$ lb is approximately $\frac{1}{4}\times\frac{1}{2.2}\text{ kg}=10^{-1}$ kg) this will enable the force acting on the bar to be found. We are given $u=0$, $s=3$ m $v=7.5 \text{ m s}^{-1}$. We want a.

$$\therefore \quad a=\frac{v^2-u^2}{2s}$$

$$\text{i.e. } a=\frac{56-0}{6}=9.3 \text{ m s}^{-2}$$

$\therefore$ The accelerating force F is, from equation 14, given by

$$F\approx 10^{-1}\times 9.3 \text{ kg m s}^{-2}$$

$$F\approx 1.0 \text{ N}$$

Question 13

The inwardly directed force, F_1, necessary to keep a body of mass m moving in a circle of radius r with a periodic time T s, i.e. with a speed $2\pi r/T$, is given by

$$F_1=mv^2/r$$

$$=m\left(\frac{2\pi r}{T}\right)^2\Big/ r$$

$$=\frac{4\pi^2 rm}{T^2}$$

Here $r=0.75$, $m=0.5$ kg, $T=0.4$ s

$$\therefore \quad F_1=\frac{4\pi^2\times 0.75\times 0.5}{0.4^2}\,\frac{m \text{ kg}}{\text{s}^2}$$

$$=\frac{39.5\times 0.75\times 0.5 \text{ N}}{0.16}$$

$$=92.5 \text{ N}$$

Question 14

Answer is 3. See section 3.5.2. Answer 1 is wrong. The newton is defined to be the force which gives 1 kg an acceleration of 1 m s^{-2}.

Question 15

Answer is 1. See section 3.5.5. Remember the difference between speed and velocity.

Question 16

Answer is 2. See section 3.5.5.

Question 17

The momentum p of the passenger in the moving car is given by,

$$p = mv$$
$$= 90 \times 60 \times 10^3/3600 \text{ kg m s}^{-1}$$
$$= 1500 \text{ kg m s}^{-1}$$

Since the passenger ends up with zero velocity p also gives the change in his momentum. In bringing the passenger to rest the belt exerts a supposedly constant force F for a time t, such that the impulse Ft equals the change in momentum p (equation 26)

$$\text{i.e. } Ft = p$$
$$\text{i.e. } F = \frac{p}{t}$$
$$= \frac{1500 \text{ kg m s}^{-1}}{0.2 \text{ s}}$$
$$\therefore \quad F = 7.5 \times 10^3 \text{ N.}$$

Figure 13 may help remind you why impulse equals change in momentum.

Question 18

From equation 31, the relativistic mass is

$$m = \frac{m_o}{\sqrt{1 - v_{im}^2/c^2}}$$

Substituting $m_o = 10^6$ kg, $v_{im} = 0.9c$, gives

$$m = \frac{10^6}{\sqrt{1 - \frac{(0.9c)^2}{c^2}}} \text{ kg}$$

$$= \frac{10^6}{\sqrt{0.19}} \text{ kg}$$

$$= 2.3 \times 10^6 \text{ kg}$$

So, from equation 32, the momentum is

$$p = mv_{im}$$
$$= 2.3 \times 10^6 \times 0.9 \times 3 \times 10^8 \text{ kg m s}^{-1}$$
$$= 6.2 \times 10^{14} \text{ kg m s}^{-1}$$

Question 19

Answer is 1. See section 3.6.2.

Question 20

Answer is 2. See section 3.6.3.